A Hyperbolic tiling problem.

**Background.**

On the Euclidean plane you can build a regular hexagon by fitting together 6 equilateral triangles around a common point which becomes the center of that hexagon. Because they fit around a common point, and there are 6 of them, these triangles have to have angle $2\pi/6$.

On a sphere every equilateral triangle has all of its angles greater than $2\pi/6$. It follows that you cannot fit six of them evenly around a ‘central’ point on the sphere. But you can fit any positive integer $n < 6$ and greater than 2 evenly around a central point. That is you can fit 5 or 4 or 3 equilateral triangles about a central point. (As you decrease the number $n$ the sides of the triangle get longer and the angles get fatter.) These ‘5,4,3’ have to do with the platonic solids.

On the hyperbolic plane every equilateral triangle has all its angles less than $2\pi/6$ so again you cannot fix six of them evenly around a central point in the hyperbolic plane. But you can fit any number $n$ GREATER than 6 evenly around a point. (!) These equilateral triangles will have three equal angles, all equal to $2\pi/n$. (As you increase the number $n$ the sides of the triangles get longer and their angles get smaller.

In the hyperboloid model the points of the hyperbolic plane are points on the “upper half of a hyperboloid of two sheets” , specifically, the plane consists of the surface $x^2 + y^2 - z^2 = -1$, $z > 0$ in Minkowski three-space. See the ‘source’ (*) from the web page as noted below to understand what these words mean.

**Problem.** Take the case $n = 8$ above and draw it in the hyperboloid model. Specifically, in this model draw a (hyperbolic!) circle of radius $R = R(8)$ centered at the point $N = (0,0,1)$. You will have to solve to find the “magic radius” $R(8)$ which is chosen precisely so that that 8 (hyperbolic!) congruent equilateral triangles fit evenly around the point N, with all their vertices besides N lying on the circle of radius $R(8)$. In this way you will have inscribed a regular octagon within your circle and these triangles will decompose the resulting regular octagon into 8 (hyperbolic!) equilateral triangles. Find a formula for the $(x, y, z)$ coordinates of these 8 vertices of this regular octagon. Make a sketch or Geogebra 5 picture of the octagon on the hyperboloid.

Hints: Observe: each angle of your equilateral triangles is $2\pi/8$.

You will need to read about the hyperboloid model. This is on our web page under ‘sources’ and it is the first hyperbolic source which is an article that appeared in the American Mathematical Monthly. You may not be able find exact values of your $x,y,z$ coordinates but instead may need to use an inverse cosh function and then approximate it.

Distance in the hyperbolic plane in this model is given by $d(A, B) = \cosh^{-1}(p(A, B))$ where $p(A, B)$ is the Minkowski inner product as described in that article, see esp eq (2,6), noting that his $x_0, x_1, x_1$ is my $z, x, y$.

**Afterward.** Now 8 of your equilateral triangles fit perfectly around any point. We can now use our triangles and the octagons they make to tile the entire hyperbolic plane with congruent octagons, each decomposed into 8 congruent equilateral triangles, in much the same way that we can tile the entire Euclidean plane with regular hexagons each of which is centrally decomposed into 6 equilateral triangles.