Comparing Sizes of functions; large and small limits.

If $p(x)$ is a polynomial with leading coefficient positive then $p(x) \to +\infty$ as $x \to 0$.

If in addition the degree of p is even then $p(x) \to +\infty$ as $x \to -\infty$, but if the degree of p is odd then $p(x) \to -\infty$ as $x \to \infty$.

Example $x^3 + x^2 \to -\infty$ as $x \to -\infty$.

Corollary [using the Intermediate Value theorem] Every odd degree polynomial has a real root.

Guiding principles for which functions are larger than as x gets large or gets small:

a) For $x > 1$ we have : $1 < x < x^2 < x^3 < ....$

b) For $0 < x < 1$ we have $x^{n+1} < x^n < \ldots < x^3 < x^2 < x < 1$.

Proofs: an inequality is true if and only if when we multiply both sides of it by a positive number it remains true.

Proof of (a) : observe that $1 < x \iff x < x^2 \iff x^2 < x^3$ etc

Proof of (b): observe that $x < 1 \iff x^2 < x \iff x^3 < x^2$ etc.

Exponentials beat polynomials

For any polynomial $p(x)$ we have that eventually $p(x) < e^x$. Moreover $p(x)/e^x \to 0$ as $x \to \infty$.

Rational Functions are quotients of polynomials: $r(x) = p(x)/q(x)$.

If $deg(p) < deg(q)$ then $r(x) \to 0$ as $x \to \infty$.

If $deg(p) > deg(q)$ then $r(x) \to \pm\infty$ as $x \to \infty$.

If $deg(p) = deg(q)$ then $r(x) \to a_d/b_d$, a finite nonzero number, where $p(x) = a_d x^d + \ldots$ and $q(x) = b_d x^2 + \ldots$.

Example $r(x) = (x^2 - 1)/(x^2 + 1)$ tends to 1 as $x \to \pm\infty$.

Here is another way to see what is going on , more directly with this example: $r(x) = (x^2+1-2)/(x^2+1) = (x^2+1)/(x^2+1) - 2/(x^2+1) = 1 + (2/(x^2+1)$ and the last term goes to 0 as $x \to \infty$.