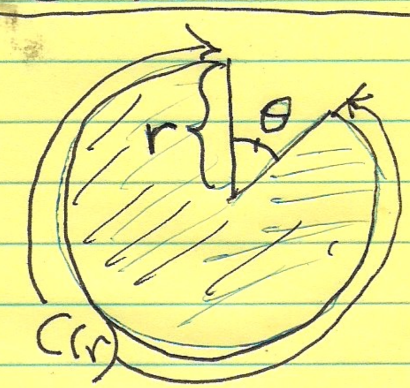
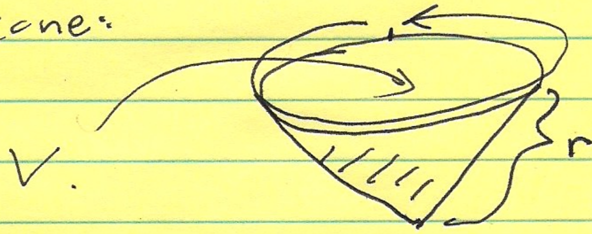
Solution to MyHWFEB11

Here's the disc with the sector cut out.



— a radius r disc, with a sector of θ radians cut out.

Here it is folded into a cone:



$C(r) = 2\pi r - r\theta$
= Circumf of top circle of cone.

We need to understand the surface area of the cone & the volume it holds as a function of r & θ

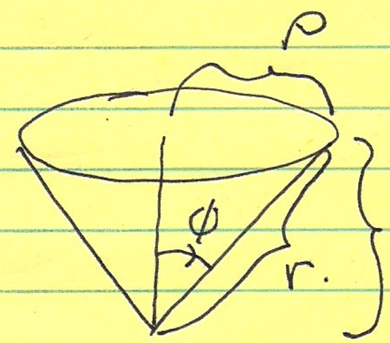
Area of disc: πr^2
Area of sector: proportional to θ.
$A(\theta) = K\theta$; $\theta = 2\pi \Rightarrow K\theta = \pi r^2$
$A(\theta) = \frac{1}{2} r^2 \theta$.

~~Similarly : circumference~~.

So $A = \pi r^2 - \frac{1}{2} r^2 \theta = \cancel{\ \ }$.

$$A = \pi r^2 \left(1 - \frac{1}{2\pi}\theta\right).$$

Volume.?

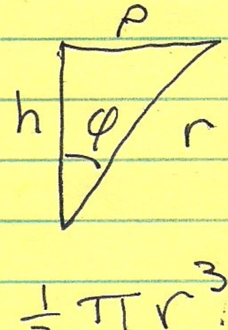


Either by looking it up, or by learning integrals ("disc method" or "washer method")

$$V = \frac{1}{3}\pi \rho^2 h.$$

Let ϕ = opening angle of cone as indicated.

Then.



: $h = r\cos\varphi$
$\rho = r\sin\varphi.$

$$\Rightarrow V = \frac{1}{3}\pi r^3 \sin^2\varphi \cos\varphi.$$

We need to understand how φ varies with θ

The circumference of the cone's top, ~~from~~ is

$$C = 2\pi\rho = 2\pi r \sin\phi, \quad \text{see p.1. fig.}$$

But it is also

$$C = 2\pi r - r\theta.$$

$$= 2\pi r\left(1 - \frac{\theta}{2\pi}\right).$$

Setting these equal, since they are!

$$2\pi r\left(1 - \frac{\theta}{2\pi}\right) = 2\pi r \sin\phi.$$

or $$\boxed{1 - \frac{\theta}{2\pi} = \sin\phi.}$$

It is helpful to note that both A & V can be expressed solely in terms of r & $\sin\phi$:

(use: $\cos\phi = \sqrt{1-\sin^2\phi}$.)

$$A = \pi r^2 \sin\phi$$

$$V = \frac{1}{3}\pi r^3 \sin^2\phi \cos\phi = \frac{1}{3}\pi r^3 \sin^2\phi \sqrt{1-\sin^2\phi}$$

A simpler choice of variables: instead of θ or ϕ: use:

$$\boxed{u = \sin\phi} \quad ; \cos\varphi = \sqrt{1-\sin^2\phi} = \sqrt{1-u^2}.$$

then

$$A = \pi r^2 u.$$

$$V = \frac{1}{3}\pi r^3 u^2 \sqrt{1-u^2}$$

We are to fix A & maximize V. It is now a nice familiar problem, with ~~real~~ nearly polynomial functions.

Solve for r: $\frac{A}{\pi u} = r^2$

so $$r^3 = (r^2)^{3/2} = \left(\frac{A}{\pi u}\right)^{3/2}.$$

$$V(u) = \frac{1}{3}\pi\left(\frac{A}{\pi}\right)^{3/2} \frac{1}{u^{3/2}} \cdot u^2 \sqrt{1-u^2}$$

$$= \frac{1}{3}\frac{\pi A^{3/2}}{\pi\sqrt{\pi}} u^{1/2}\sqrt{1-u^2}.$$

Brute force:
~~Bone head~~ way

Set
$$V'(u) = 0$$

get:

$$0 = C\left(\frac{1}{2}u^{-1/2}\sqrt{1-u^2} + u^{1/2}\frac{1}{2}(1-u^2)^{-1/2}\cdot(-2u)\right)$$

$$= C\left(\frac{\sqrt{1-u^2}}{2\sqrt{u}} - \frac{u^{3/2}}{\sqrt{1-u^2}}\right)$$

where $C = \frac{1}{3}\frac{A^{3/2}}{\pi}$ & is not important!

we get:

$$\frac{\sqrt{1-u^2}}{2\sqrt{u}} = \frac{u^{3/2}}{\sqrt{1-u^2}}.$$

both sides: $\left(\times\sqrt{1-u^2}\ \sqrt{u}\right)$

$$\frac{1-u^2}{2} = u^2$$

$$\frac{1}{2} - \frac{u^2}{2} = u^2$$

$$\frac{1}{2} = \frac{3}{2}u^2 \Rightarrow u^2 = \frac{1}{3}$$

$$u = \sqrt{\frac{1}{3}}.$$

Now go back!

remember

$$u = \left(1 - \frac{\theta}{2\pi}\right)$$

so

$$\frac{1}{\sqrt{3}} = 1 - \frac{\theta}{2\pi}$$

or

$$\frac{\theta}{2\pi} = 1 - \frac{1}{\sqrt{3}} \simeq .4226.$$

Cut out about 42%.
of the whole disc
to make this optimal
cone.

alternative to p. 5 computation

<u>Slicker</u> but <u>more theory</u> way

$5\frac{1}{2}$/7

At this point we could just maximize $V(u)$ by setting $V'(u)=0$.

But pause a second...
observe: u_* minimizes $V(u)$ if & only if u_* minimizes $V(u)^2$.

[Why? <u>Lemma</u>: let F be any *strictly* <u>monotonic</u> <u>increasing</u> function of a nonnegative variable. Then for a nonnegative function $V(u)$, u_* maximizes $V(u)$ if and only if u_* maximizes $F(V(u))$.]

$$V(u)^2 = C^2 u(1-u^2), \quad C = \frac{1}{3}\frac{A^{3/2}}{\sqrt{\pi}}.$$

$$\frac{d}{du}(V(u))^2 = \frac{d}{du}C^2(u-u^3)$$

$$= C^2(1-3u^2).$$

So $$\frac{d}{du}(V(u))^2 = 0 \iff 1-3u^2=0$$

$$\iff u^2 = \frac{1}{3}.$$