Example 1. Let a and b be the diagonal and the side of a regular pentagon. Then a and b are non-commensurable.

Proof. The proof is to show that the algorithm of alternating measurement, applying to such a pair $\{a, b\}$ will never end!

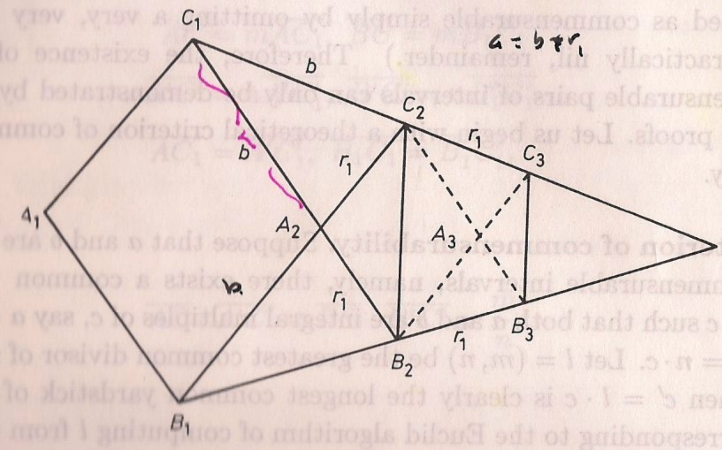

Fig. 3

As indicated in Fig. 3, $A_1B_1B_2C_2C_1$ is a regular pentagon whose side length and diagonal length are b and a respectively and its five inner angles are all equal to $\frac{3\pi}{5}$. $\Delta C_1B_2C_2$ is an isosceles triangle, thus

$$\angle C_1B_2C_2 = \angle B_2C_1C_2 = \frac{1}{2}\left(\pi - \frac{3\pi}{5}\right) = \frac{\pi}{5} \tag{9}$$

and the same reason shows that $\angle B_2C_2B_1 = \frac{\pi}{5}$. Therefore $\Delta A_2B_2C_2$ is also an isosceles triangle and $\angle B_2A_2C_2 = \pi - \frac{\pi}{5} - \frac{\pi}{5} = \frac{3\pi}{5}$. Moreover,

$$\angle C_1A_2C_2 = \pi - \frac{3\pi}{5} = \frac{2\pi}{5}, \ \angle C_1C_2A_2 = \frac{3\pi}{5} - \frac{\pi}{5} = \frac{2\pi}{5} \tag{10}$$

and hence $\Delta C_1A_2C_2$ is also an isosceles triangle.

Thus

$$a = \overline{C_1B_2} = \overline{C_1A_2} + \overline{A_2B_2} = b + r_1, \ r_1 = \overline{A_2B_2} = \overline{A_2C_2}. \tag{11}$$

Extend $\overline{B_1B_2}$ (resp. $\overline{C_1C_2}$) to B_3 (resp. C_3) such that $\overline{B_2B_3} = \overline{C_2C_3} = r_1$. Then, as indicated in Fig. 3, the pentagon $A_2B_2B_3C_3C_2$ is again a regular one! (The proof of this fact is a simple exercise.) Moreover, its diagonal length is b while its side length is r_1. Therefore, as one proceeds to measure b by r_1 as the yardstick, the geometric situation is exactly the same as before, namely, the remainder is the difference between the diagonal and the side of a regular pentagon. Thus

$$b = r_1 + r_2, r_1 = r_2 + r_3, \ldots, r_{k-1} = r_k + r_{k+1}, \ldots \tag{12}$$

where the pair $\{r_{k-1}, r_k\}$ are always the diagonal and the side of a regular pentagon! Of course, this algorithm can never end, although the size of the k-th regular pentagon gets smaller and smaller. This proves that $\{a, b\}$ are *non-commensurable*!

Example 2. After he discovered the above astonishing example of non-commensurable pair of intervals by the above simple ingenious proof, Hippasus naturally proceeded to analyze the commensurability problem between the diagonal and the side of a square, say a' and b'. As indicated by Fig. 4, it is not difficult to show that the algebraic relationships among the remainders of the algorithm of alternating measurements are as follows, namely

$$a' = b' + r_1, b' = 2r_1 + r_2, r_1 = 2r_2 + r_3, \ldots, r_{k-1} = 2r_k + r_{k+1}, \ldots \tag{13}$$

Therefore, the geometric situations from the second one onward are all the same and hence this algorithm can never end! Thus $\{a', b'\}$ is again a non-commensurable pair.

[We leave the geometric proof of (13) as an exercise.]

Historical remarks

(i) The above discovery of non-commensurable pairs by Hippasus is a monumental milestone in the entire human civilization of rational mind. However, to his fellow Pythagoreans and contemporary

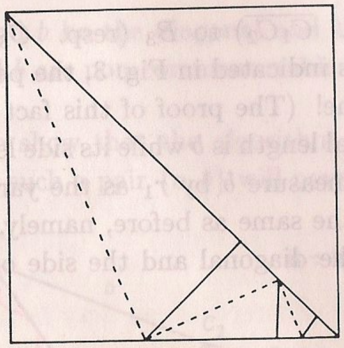

Fig. 4

geometers, this was a gigantic "*geoquake*" which rocked the whole foundation of quantitative geometry. The proofs of the area formula of rectangle and the similar triangle theorem that they prided were *no longer* complete proofs covering full generality, but rather, they were merely proofs for the *special commensurable case* only.

(ii) The historical record of this great event is unfortunately lost. However, according to some indirect sources, the following story may roughly serve as an account of what was happening to Hippasus and his great discovery: The initial reaction of his fellow Pythagoreans were shock and denial and, in order to avoid the unbearable embarrassment of public disgrace, they decided to cover it up and vowed to keep it as a secret. However, such a covering up of fundamental truth, eventually, became unbearable for the scholar Hippasus and he somehow leaked the truth of his great discovery to the outsiders (which, by the way, were often referred by the Pythagoreans simply as "the unworthies"). This made his fellow Pythagoreans furious and they condemned him to death! Naturally, he fled away. But unfortunately, the Pythagoreans were eventually able to track him down on a merchant ship in the Mediterranean and they pushed him overboard. Thus, a great hero of human civilization died for the truth. One might add here that the above story should probably be labelled as "*the first Pentagon Paper*".

(iii) Actually, his fellow Pythagoreans should be proud of such a monumental discovery by their school. Moreover, although their first attempt in building a foundation of quantitative geometry was not as perfect as they thought, it was still a major step forward and an impressive achievement by itself. Therefore, the proper reaction should be to celebrate the new discovery of their colleague, admitting the inadequacy of their proofs based upon the false axiom of universal validity of commensurability and then resolved to work for the proofs of the remaining non-commensurable case. Of course, such proofs were by no means easy to find and they naturally became the major challenge to the entire community of Greek geometers of that time. The task of rebuilding a solid foundation of quantitative geometry was finally succeeded by Eudoxus (408–355 B.C.) and his successful story is naturally our next topic of discussion.

1.3. *Eudoxian principle, the origin of the methodology of approximation*

Let us begin with some analysis of the task that Eudoxus and his contemporary were facing.

Analysis

1. In the case that two intervals a and b are *commensurable*, the ratio between their lengths has a clear simple meaning and it is a rational number. However, in the case that two intervals a and b are *non-commensurable* (such as the case of Examples 1 and 2), the meaning of the ratio between them is something yet to be defined and it is definitely *not* a rational number.

2. Although the meaning of the ratio between two non-commensurable pairs of intervals a and b is still undefined, the meaning of inequality between such a yet to be defined ratio and a given rational number $\frac{m}{n}$ such as

$$a : b > \frac{m}{n} \text{ or } a : b < \frac{m}{n}$$

is, in fact, quite clear, namely

(i) $a : b > \frac{m}{n}$ if $n \cdot a$ is longer than $m \cdot b$,

(ii) $a : b < \frac{m}{n}$ if $n \cdot a$ is shorter than $m \cdot b$.

3. Suppose that a and b are a given pair of non-commensurable intervals. Then, to a given natural number n, one may first subdivide b into n equal parts and then use $\frac{1}{n} \cdot b$ as the yardstick to measure a, thus obtaining an m such that $m \cdot \frac{1}{n} b$ is shorter than a while $(m+1)\frac{1}{n} b$ is longer than a, namely

$$\frac{m}{n} < a : b < \frac{m+1}{n}. \tag{14}$$

Therefore, the difference between $a : b$ and $\frac{m}{n}$ (resp. $\frac{m+1}{n}$) is, of course, less than $\frac{1}{n}$, although the meaning of $a : b$ is yet to be defined. By choosing n sufficiently large, the above difference can be *as small as one wishes*!

4. Suppose that $\{a, b\}$ and $\{a', b'\}$ are two given pairs of non-commensurable intervals. How do we compare their ratios $a : b$ and $a' : b'$? Suppose that $a : b < a' : b'$. Then one may choose n sufficiently large such that $\frac{1}{n}$ is smaller than the difference between the above two ratios (whatever the meaning of the difference of such two ratios may eventually be defined to be). Let m be such an integer that $\frac{m}{n} < a : b < \frac{m+1}{n}$. Then $\frac{m+1}{n}$ must be smaller than $a' : b'$, namely

$$a : b < \frac{m+1}{n} < a' : b'. \tag{15}$$

The above analysis naturally leads to the following definition enunciated by Eudoxus, namely

Eudoxian principle. Let $\{a, b\}$ and $\{a', b'\}$ be two pairs of non-commensurable intervals. If there exists a fraction $\frac{m}{n}$ such that

$$a : b < \frac{m}{n} < a' : b' \text{ (resp. } a : b > \frac{m}{n} > a' : b')$$

then $a' : b'$ is *defined* to be larger (resp. smaller) than $a : b$. On the other hand, if any rational $\frac{m}{n}$ which is larger (resp. smaller) than

$a : b$ is also larger (resp. smaller) than $a' : b'$, then $a' : b'$ is *defined* to be equal to $a : b$, namely, a necessary and sufficient condition of $a : b = a' : b'$ is that

$$na \begin{Bmatrix} > \\ < \end{Bmatrix} m \cdot b \Leftrightarrow na' \begin{Bmatrix} > \\ < \end{Bmatrix} mb' \tag{16}$$

for all m and n.

The above *criterion* of the equality between the ratios of two pairs of non-commensurable intervals is undoubtedly correct. However, such a criterion needs to verify that $a : b$ and $a' : b'$ have *identical inequality relationships with all rational numbers* in order to establish the *equality* between them. That means one has to check *infinitely many* inequalities in order to obtain a single equality. One cannot help but wonder about the usefulness of such a criterion. The first major application of the above Eudoxian principle is to provide a firm foundation of geometry. The following proof of the similar triangle theorem for the remaining case of non-commensurable ones is a typical example of such an application.

Example 3. A proof of similar triangle theorem for the non-commensurable case by Eudoxian principle. Let $\frac{m}{n}$ be an arbitrary fraction number. We shall show that

$$\overline{AB} : \overline{A'B'} \begin{Bmatrix} > \\ < \end{Bmatrix} \frac{m}{n} \Rightarrow \overline{AC} : \overline{A'C'} \text{ and } \overline{BC} : \overline{B'C'} \begin{Bmatrix} > \\ < \end{Bmatrix} \frac{m}{n}. \tag{17}$$

Set B_1 to be the point on AB such that $AB_1 : \overline{A'B'} = \frac{m}{n}$, and C_1 to be the point on AC such that $B_1C_1 // BC$. Then

$$\angle B_1 = \angle B = \angle B', \ \angle C_1 = \angle C = \angle C'. \tag{18}$$

Therefore, by the proven commensurable case of the theorem

$$\overline{AC_1} : \overline{A'C'} = \overline{B_1C_1} : \overline{B'C'} = \frac{m}{n}. \tag{19}$$

If $\overline{AB} : \overline{A'B'} > \frac{m}{n}$ (resp. $< \frac{m}{n}$), then