THEOREM 2 Limit Laws for Sequences Assume that $\{a_n\}$ and $\{b_n\}$ are convergent sequences with

$$\lim_{n\to\infty} a_n = L, \qquad \lim_{n\to\infty} b_n = M$$

Then:

(i) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$

(ii) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$

(iii) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{L}{M}$ if $M \neq 0$

(iv) $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cL$ for any constant c

THEOREM 3 Squeeze Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that for some number M,

$$b_n \le a_n \le c_n \quad \text{for } n > M \quad \text{and} \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$$

Then $\lim_{n\to\infty} a_n = L$.

EXAMPLE 8 Show that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Solution We have

$$-|a_n| \le a_n \le |a_n|$$

By hypothesis, $\lim_{n\to\infty} |a_n| = 0$, and thus also $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0$. Therefore, we can apply the Squeeze Theorem to conclude that $\lim_{n\to\infty} a_n = 0$. ■

EXAMPLE 9 Geometric Sequences with $r < 0$ Prove that for $c \neq 0$,

$$\lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if} \quad -1 < r < 0 \\ \text{diverges} & \text{if} \quad r \le -1 \end{cases}$$

Solution If $-1 < r < 0$, then $0 < |r| < 1$ and $\lim_{n\to\infty} |cr^n| = 0$ by Example 7. Thus,

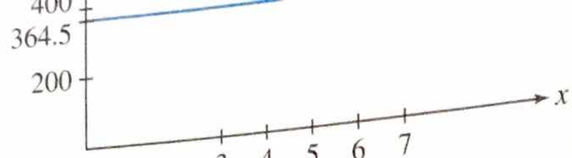

FIGURE 7 The sequence and the function approach the same limit.

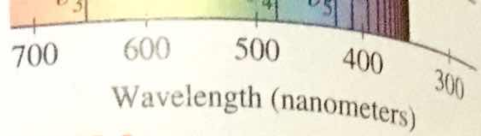

FIGURE 8

EXAMPLE 6 **Balmer Wavelengths** Calculate the limit of the Balmer wavelengths $b_n = \dfrac{364.5n^2}{n^2 - 4}$ in nanometers, where $n \ge 3$.

Solution Apply Theorem 1 with $f(x) = \dfrac{364.5x^2}{x^2 - 4}$:

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} \frac{364.5x^2}{x^2 - 4} = \lim_{x\to\infty} \frac{364.5x^2 \frac{1}{x^2}}{(x^2 - 4)\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{364.5}{1 - 4/x^2} = \frac{364.5}{\lim\limits_{x\to\infty} (1 - 4/x^2)} = 364.5 \text{ nm}$$

A **geometric sequence** is a sequence $a_n = cr^n$, where c and r are nonzero constants. Each term is r times the previous term; that is, $a_n/a_{n-1} = r$. The number r is called the **common ratio**. For instance, if $r = 3$ and $c = 2$, we obtain the sequence (starting at $n = 0$)

$$2, \quad 2 \cdot 3, \quad 2 \cdot 3^2, \quad 2 \cdot 3^3, \quad 2 \cdot 3^4, \quad 2 \cdot 3^5, \quad \dots$$

In the next example, we determine when a geometric series converges. Recall that $\{a_n\}$ **diverges to** ∞ if the terms a_n increase beyond all bounds (Figure 9); that is,

$$\lim_{n\to\infty} a_n = \infty \quad \text{if, for every number } N,\ a_n > N \text{ for all sufficiently large } n$$

We define $\lim\limits_{n\to\infty} a_n = -\infty$ similarly.

EXAMPLE 7 **Geometric Sequences with $r \ge 0$** Prove that for $r \ge 0$ and $c > 0$,

$$\lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if} \quad 0 \le r < 1 \\ c & \text{if} \quad r = 1 \\ \infty & \text{if} \quad r > 1 \end{cases}$$

Solution Set $f(r) = cr^x$. If $0 \le r < 1$, then (Figure 10)

$$\lim_{n\to\infty} cr^n = \lim_{x\to\infty} f(x) = c \lim_{x\to\infty} r^x = 0$$

If $r > 1$, then both $f(x)$ and the sequence $\{cr^n\}$ diverge to ∞ (because $c > 0$) (Figure 9). If $r = 1$, then $cr^n = c$ for all n, and the limit is c.

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$n! =$

For example definition,

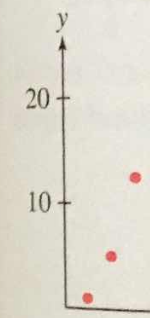

FIGURE 11