March 2.

Today:
- Lecture Final problem 4 (L'Hôpital

- Missing Problem: from HW / Limits of sequence
$a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$
& inductively

Prove: The limit exists.
& Estimate it.

On the way:
More on e^x
on variants of e^x

2nd order Taylor polynomials
$\cos(r_n x) - \cos(wx)$

More on " limits.
→ Pinching theorems (Monotone theorem
Beginning Induction.

Overall Plan: 1 problem from Review per lecture.

1 New problem, till end.

Mon Mar 12: Special.

Warm up:

$$\lim_{t\to 0} \frac{e^t - 1}{t}.$$

OK $\lim_{t\to 0} \frac{e^t + e^{-t} - 2}{t^2}$

$f(t) = e^t + e^{-t} - 2$

$g(t) = t^2.$

<u>Way 1</u>: $f(0) = 1 + 1 - 2 = 0$

$g(0) = 0^2 = 0$

of form $\frac{0}{0}$. So:

$f'(t) = e^t - e^{-t}$

$g'(t) = 2t.$

$f'(0) = 1 - 1 = 0$

$g'(0) = 2 \cdot 0 = 0$

Still of form $0/0$

L'H 2

$f''(t) = e^t - (-)e^{-t} = e^t + e^{-t}$

$g''(t) = 2.$

$f''(0) = 1+1 = 2$

So

$$\lim_{t \to 0} \frac{f(t)}{g(t)} = \frac{f''(0)}{g''(0)} = \frac{2}{2} = 1.$$

$O(t^4)$

<u>Way 2</u>:

$$e^t = 1 + t + \frac{t^2}{2} + \cdots$$

$$e^{-t} = 1 - t + \frac{t^2}{2} + \cdots$$

so

$$e^t + e^{-t} = 2 + 0 + t^2 + \cdots$$

& $$\frac{e^t + e^{-t} - 2}{t^2} = \frac{2 + t^2 + \cdots - 2}{t^2}$$

$$= \frac{t^2 + O(t^4)}{t^2} = 1 + O(t^2)$$

& $$\lim_{t \to 0} \frac{e^t + e^{-t} - 2}{t^2} = \lim_{t \to 0} 1 + O(t^2) = 1.$$

cosh & sinh ? L'H3

& sin, cos

$y'' = y$ vs $\cancel{y}'' = -y$

Sol'ns:

$A = ae^{x} + be^{-x}$ vs $a\sin x + b\cos x$.

More generally:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0.$$

Try $y(x) = e^{rx}$.

$$\frac{dy}{dx} = re^{rx}.$$

$$\frac{d^2y}{dx^2} = r^2e^{rx}.$$

so

$$P\left(\frac{d}{dx}\right)e^{rx} = P(r)e^{rx}.$$

eg: $\left(\frac{d^2}{dx^2} - 1\right)y = 0$

$r^2 =$

The $\sqrt{2+\sqrt{2+\cdots}}$ problem.

$1 < \sqrt{2} < 2$

we set

$a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

etc so

$$a_{n+1} = \sqrt{2+\sqrt{a_n}}.$$

Strategy:

Observe

$a_1 < a_2 < a_3$

"monotone"

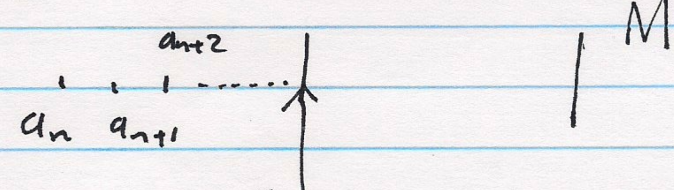
To show 1) $a_n < a_{n+1}$

2) For <u>some</u> number M

"Bounded" & all n's

$a_n < M$

Then:



Use : Any <u>Bounded</u> $^{(M)}$ monotone $\uparrow$ sequence of numbers has a unique limit

How to get bound?

Try $M = 2$ 1st.

since

$a_1 < 2, \quad a_2 = \sqrt{2+\sqrt{2}} \approx \sqrt{3.14} < 2$

if $a_n < 2$ what about a_{n+1}?

$a_n^2 < 4$

$2 + \sqrt{a_n} < 2 + \sqrt{2} < 4$

so

$\sqrt{2+\sqrt{a_n}} < \sqrt{4} = 2.$

We have a proof by induction!

We know $a_1 < 2$

we have proved that if

$a_n < 2$

then

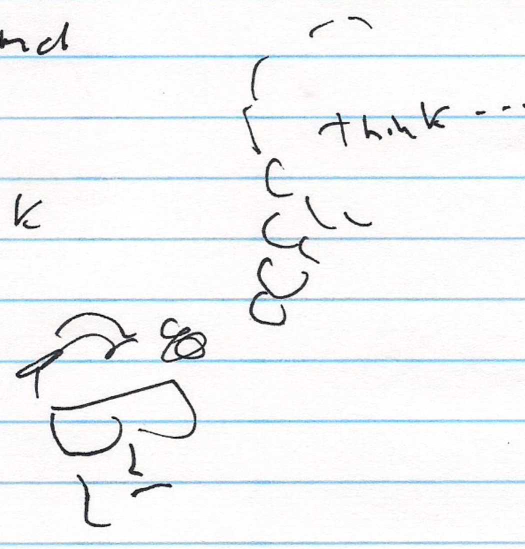
$a_{n+1} < 2.$

How to show increasing?

again by induction.
(although it seems obvious.

$a_1 < a_2$ since $2 < 2+\sqrt{2}$
so $\sqrt{2} < \sqrt{2+\sqrt{2}}$.

Say we've shown
$$a_n < a_{n+1}$$
well:
$$a_{n+2} = \sqrt{2+\sqrt{a_{n+1}}}$$
$$a_{n+2}^2 = 2+\sqrt{a_{n+1}}$$
$$\vdots$$
Dead end
$$\vdots$$
stuck



limits 3A

scratch

Anyone... ??

Hmmm.

$\sqrt{2+\sqrt{a_n}} \overset{?}{>} a_n.$

$2+\sqrt{a_n} > a_n^2.$

$2 > a_n^{3/2}$

is $a_n < 2^{2/3}.$ ← would bring us to victory!

Go back: ....

True?

IF --> By induc. $a_n < 2^{2/3} = \frac{2}{\sqrt[3]{2}}.$

a_1 ✓ $\sqrt{2} < 2^{2/3}.$

$a_{n+1} \overset{?}{=} \sqrt{2+\sqrt{a_n}}$

$a_{n+1}^2 = 2+\sqrt{a_n} < 2+2^{1/3} = 2^{4/3}.$

$a_n < 2^{2/3}.$

Hah!

limits 4

Go back

Prove, by induction on n that

$$a_n < 2^{2/3}.$$

then, $a_n \nearrow$ directly since.

$$a_{n+1} = \sqrt{2+\sqrt{a_n}}.$$

$$a_{n+1}^2 = 2+\sqrt{a_n}$$

&

$$a_{n+1}^2 > a_n \iff 2+\sqrt{a_n} > a_n^2$$

$$\iff 2 > \frac{a_n^2}{\sqrt{a_n}} = a_n^{3/2}$$

$$\iff 2^{2/3} > a_n.$$
