The derivative as the magnified graph.

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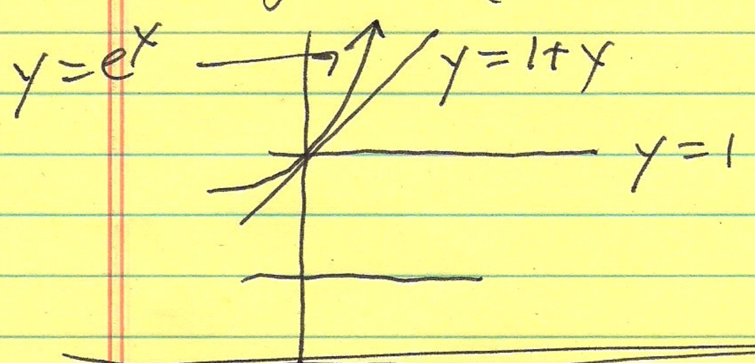
last time: Near to $\cancel{x} = 0$ graph of e^x:



Algebro-Analytic def of derivative:

$$f'(x_0) = \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon}.$$

So:

$$\lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon(X - x_0)) - f(x_0)}{\varepsilon} = f'(x_0)(X - x_0)$$

Since: for $X - x_0 \neq 0$, $\varepsilon(X - x_0) \neq 0$.

& $\dfrac{f(x_0 + \varepsilon(X - x_0)) - f(x_0)}{\varepsilon(X - x_0)} \cdot (X - x_0)$

$$= \frac{f(x_0 + \varepsilon(X - x_0)) - f(x_0)}{\varepsilon}.$$

BL 2

& $\varepsilon \to 0 \iff \varepsilon(X - x_0) \to 0.$

so:

$$\lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon(X - x_0)) - f(x_0)}{\varepsilon(X - x_0)}$$

$$= \lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon}.$$

To show:

in the limit of infinite magnification or "blow-up" the graph tends to the graph of its linearization ie: a line with slope the derivative.

First at $X = 0$, assuming $f(0) = 0$.

BL 3

¿ How does a smart phone algebraically magnify or shrink?

(Pixels: $(x,y) \in \mathbb{R}^2 \cap \mathbb{Z}^2$)

By a factor of 3?

$$(x,y) \longmapsto (3x, 3y)$$

By λ: $(x,y) \longmapsto (\lambda x, \lambda y)$.

This operation is scaling by λ, with center origin.

Magnifying a graph:

$$\Gamma_f = \{(x,y) : y = f(x),\ x \in \mathbb{R}\}.$$

$\equiv$ graph of $f : \mathbb{R} \to \mathbb{R}$.

Scaled graph:

$$\lambda \Gamma_f = \{(\lambda x, \lambda y) : y = f(x);\ x \in \mathbb{R}\}$$

$$= \{(X, Y) : Y = \lambda f\left(\frac{X}{\lambda}\right) : X \in \mathbb{R}\}$$

since if $X = \lambda x$, $f(x) = f\left(\frac{X}{\lambda}\right)$.

BL 4

Now set $\varepsilon = \frac{1}{\lambda}$, $\lambda = \frac{1}{\varepsilon}$.

~~opt~~ & assume:
$f(0) = 0$; & $f'(0)$ exists

$$\frac{1}{\varepsilon}\Gamma_f = \{(X, Y) : Y = \frac{1}{\varepsilon} f(\varepsilon X) : X \in \mathbb{R}\}$$

By def. of derivative, as $\varepsilon \to 0$, for frozen X.

$$\frac{1}{\varepsilon} f(\varepsilon X) \longrightarrow f'(0) \cdot X$$

The magnified graph tends to the tangent line.

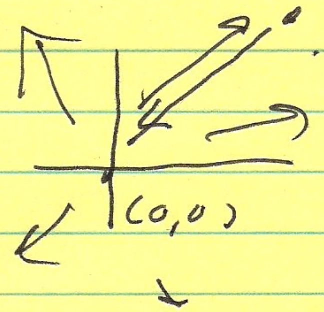
Flat earth

BL 5

What about at $(x_0, y_0) \neq (0,0)$

To magnify about a different center:

$\vec{c} = (x_0, y_0)$



translate center to origin;
magnify there,
translate back:

$(x,y) \longmapsto (x-x_0, y-y_0) \to (\lambda(x-x_0),$

$\to (\lambda(x-x_0), \lambda(y-y_0))$

$\to (\lambda(x-x_0)+x_0, \lambda(y-y_0)+y_0)$

Magnification by λ, ← factor.
center (x_0, y_0)

$$D_\lambda^{(x_0,y_0)}(x,y) = (\lambda(x-x_0)+x_0, \lambda(y-y_0)+y_0)$$

BL 6

Apply to Γ_f where

$(x_0, y_0) \in \Gamma_\lambda$, so $y_0 = f(x_0)$.

$$D_\lambda^{x_0,y_0} \Gamma_f$$

$$= \{(\underbrace{\lambda(x-x_0)+x_0}_{\text{new } X}, \lambda(y-y_0)+y_0 : y = f(x), x \in \mathbb{R}\}.$$

$y = f(x)$;

$$X = \lambda(x-x_0)+x_0$$

$$X - x_0 = \lambda(x-x_0).$$

$$\frac{1}{\lambda}(X-x_0) = x - x_0.$$

$$\frac{1}{\lambda}(X-x_0)+x_0 = x.$$

so:

$$Y = \lambda\{f(\tfrac{1}{\lambda}(X-x_0)+x_0) - f(x_0)\} + f(x_0).$$

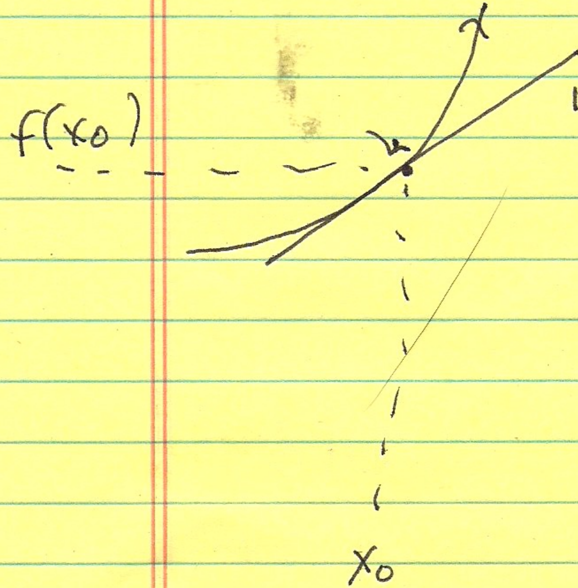
$$= \frac{1}{\varepsilon}[f(x_0+\varepsilon(X-x_0)) - f(x_0)\} + f(x_0).$$

In limit $\varepsilon \to 0$:

$$Y = f'(x_0)(X-x_0) + f(x_0).$$

Scaled graph $\to$ graph of linear approx at x_0.

Review; Meaning of 2nd derivative.

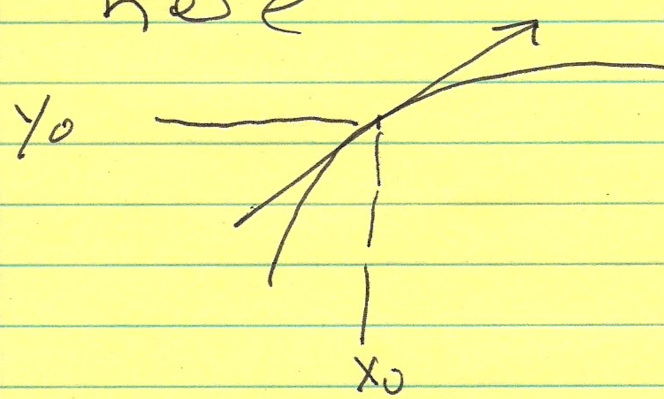


linearization : slope: $f'(x_0)$

In this picture what is te <u>sign</u> of $f''(x_0)$

here

sign $f''(x_0)$



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Here:

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