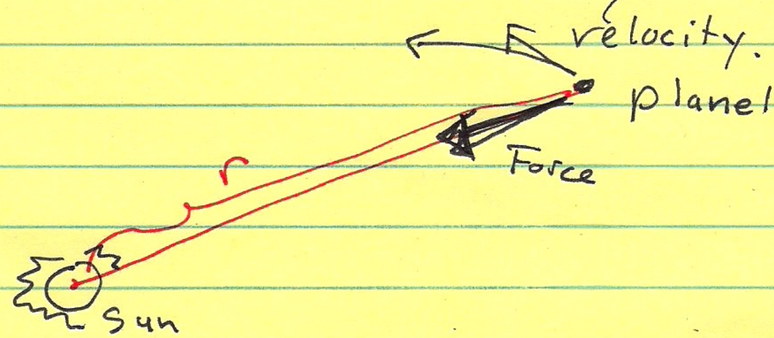
K1

Beginnings of Modern Physics, Astronomy ....

Newton : ~~to~~ sets up laws of mechanics as a differential eqn; derives Kepler's laws of planetary motion.



Represent location of planet as a point in the Cartesian (x,y)-plane (plane of 'ecliptic') with sun at origin.

Force law: directed in to sun.

$$\text{strength} \propto \text{mass}_{\text{planet}} \times \frac{1}{r^2}.$$

$$\vec{F} = m\vec{a}$$

Force. mass acceleration

K2

Newton used $\cdot$ for $\frac{d}{dt}$

& imagined all variables depending on t

so ~~acce~~ if $\vec{q}$ = position

$\dot{\vec{q}}$ = velocity

$\ddot{\vec{q}}$ = acceleration.

Now:

$\vec{q} = (x, y)$; $\dot{q} = (\dot{x}, \dot{y})$

so equs:

$$\left.\begin{aligned} m\ddot{x} &= -\frac{mGx}{r^3} \\ m\ddot{y} &= -\frac{mGy}{r^3} \end{aligned}\right\}$$ why not $\frac{1}{r^3}$.?

at (x,y) the unit inward vector is ~~th~~ $\left(\frac{x}{r}, \frac{y}{r}\right)$.

Why?

case $(x,y) = (x,0)$, $x>0$.

unit vector, pointing in??

or if

$$\vec{r} = (x,y)$$
$$r = \sqrt{x^2+y^2}.$$

constant.

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\mu}{r^2}\left(\frac{\vec{r}}{r}\right).$$

unit vector

How to solve??

Some simpler ones 1st

Motion along a line.

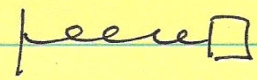
$$\ddot{x} = 0$$

(?)

K4

$$\ddot{x} = -x$$

<u>Solve</u>!



Hooke:

$$m\ddot{x} = -Kx.$$

Rewrite "Kepler problem"

$$\boxed{\ddot{\vec{r}} = -\mu f(r)\left(\frac{\vec{r}}{r}\right)} \quad ; \quad f(r) = \frac{1}{r^2}.$$

$\vec{r}$ ← a vector

$$\dot{\vec{r}}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \text{velocity}$$

<u>Theorem</u>: If $\vec{r}(t) = (x(t), y(t))$ is differentiable ~~term~~

$$\dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t))$$

$\uparrow \quad \nearrow$

usual 'single variable' derivative.

<u>or</u>

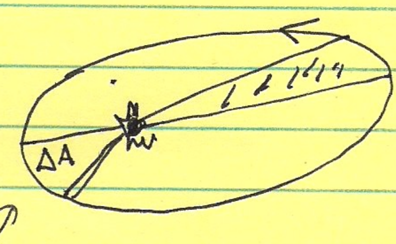
$$\begin{cases} \ddot{x} = -\mu f(r) \dfrac{x}{r} \\ \ddot{y} = -\mu f(r) \dfrac{y}{r} \end{cases}$$

Goal derive Kepler's 3 laws of planetary motion from the 'Kepler problem ODE)

K1 : planets move in ellipses with the sun being one focus

K2 : as the planet moves it sweeps out equal areas in equal times

$\Delta A = c \Delta t$



if Δt the same then ΔA the same

K3 : $(\text{orbit size})^3 = K(\text{planetary year})^2$

if a measures linear size of orbit & T " year (period)

$$\boxed{a^3 = KT^2}$$

spacings & years of planets.

K2 $\Longleftrightarrow$

Angular momentum: is

$$x\dot{y} - y\dot{x}$$

is "conserved": constant along solutions

Proof

$$\frac{d}{dt}(x\dot{y} - y\dot{x}) = \dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x}$$

$$= \underbrace{\dot{x}\dot{y} - \dot{y}\dot{x}}_{0} + x\ddot{y} - y\ddot{x}$$

But

$$\ddot{x} = -\mu f(r)\frac{x}{r}$$

$$\ddot{y} = -\mu f(r)\frac{y}{r}.$$

so $$x\ddot{y} - y\ddot{x} = -\mu\frac{f(r)}{r}xy + \mu\frac{f(r)}{r}xy$$

$$= 0.$$

Now use polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

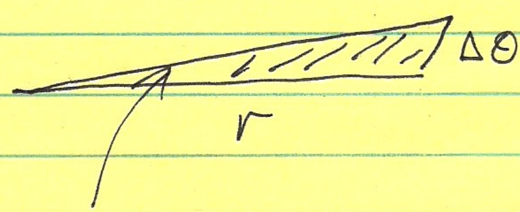
K7

Compute:

$$x\dot{y} - y\dot{x} = r^2\dot{\theta}$$

Set $J = x\dot{y} - y\dot{x}$

& use knowledge of circles.



area: $\frac{1}{2} r(r\Delta\theta) = \frac{1}{2} r^2 \Delta\theta.$

Can prove:

Area swept out in time t is

$$\int \frac{1}{2} r^2 \, d\theta = \int \frac{1}{2} r(t)^2 \frac{d\theta}{dt} \, dt$$

$$= \int \frac{1}{2} r(t)^2 \dot{\theta} \, dt$$

$$\Delta A(t) = \frac{1}{2} J \Delta t$$

$$\boxed{\Delta A = K \Delta t.} \qquad K = \frac{1}{2} J$$

or

K $7\frac{1}{2}$

So:

$$\frac{dA}{dt} = r^2\dot{\theta} = \text{const}$$

or $A(t) = c(t-t_0)$.

"equal areas in equal times"

$$\frac{\Delta A}{\Delta t} = c.$$

or

$$\Delta A \propto \Delta t.$$

Means

$$\Delta A = K\Delta t.$$

$\frac{1}{2}$ angular momentum.

Establishing K2
for any central
force law.

ie arbitrary $f(r)$.

eg: $\ddot{x} = -\mu \frac{x}{r^5}$

$\ddot{y} = -\mu \frac{y}{r^5}$.

Partial justification of K3
Now special to $f(r) = \frac{1}{r^2}$.

1st: Suppose

$\Rightarrow \vec{r}(t) = (R_0 \cos\omega t, R_0 \sin\omega t)$

[~~circular~~ uniform circular motion]

Compute $\ddot{\vec{r}}(t)$.

K3

K9.

Keplers 3rd Law
special case of circles.

$c(t) = (R_0 \cos \omega t, R_0 \sin \omega t)$.

$\ddot{c} = -\omega^2 R_0 \left(\frac{\vec{c}}{|\vec{c}|}\right)$

$= - \frac{\mu}{r^2} \left(\frac{\vec{c}}{|c|}\right)$.

$\Leftrightarrow \cancel{A}\ \omega^2 R_0 = \frac{\mu}{R_0^2}$

or

$\omega^2 R_0^3 = \mu$.

But $\omega = \frac{2\pi}{T}$.

$\Rightarrow R_0^3 = c T^2$

$c = \frac{\mu}{(2\pi)^2}$.