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CHAPTER **1**

THE CENTRAL FORCE PROBLEM

1. FORMULATION OF THE PROBLEM

Celestial mechanics begins with the central force problem: to describe the motion of a particle of mass m which is attracted to a fixed center O by a force $mf(r)$ which is proportional to the mass and depends only on the distance r between the particle and O. The function f will be called a *law of attraction*. It is assumed to be continuous for $0 < r < \infty$.

Mathematically, the problem is easy to formulate. Indicate the position of the mass by the vector $\mathbf{r}$ directed from O. According to Newton's second law, the motion of the particle is governed by the equation

$$m\ddot{\mathbf{r}} = -mf(r)r^{-1}\mathbf{r},$$

where $r^{-1}\mathbf{r}$ is a unit vector directed to the position of the particle. If $\mathbf{v}$ denotes the velocity vector $\dot{\mathbf{r}}$, the equation can be written as the pair

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = -f(r)r^{-1}\mathbf{r}. \tag{1.1}$$

Observe that the value of m is irrelevant to the equations of motion. The problem is now this: to study the properties of pairs of vector-valued functions $\mathbf{r}(t)$, $\mathbf{v}(t)$ which simultaneously satisfy the Eqs. (1.1) over an interval of time.

The special case when the law of attraction is Newton's law of gravitation is the most important. In this case $f(r) = \mu r^{-2}$, where μ is a positive constant depending only on the units chosen and on the particular source of attraction. The Eqs. (1.1) become

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = -\mu r^{-3}\mathbf{r}. \tag{1.2}$$

2. THE CONSERVATION OF ANGULAR MOMENTUM: KEPLER'S SECOND LAW

Let us now assume that (1.1) is satisfied for some interval of time by the pair of functions $\mathbf{r}(t)$, $\mathbf{v}(t)$ which we write simply as $\mathbf{r}$, $\mathbf{v}$. From the second equation of the pair we conclude that

$$\mathbf{r} \times \dot{\mathbf{v}} = -f(r)r^{-1}(\mathbf{r} \times \mathbf{r}) = 0,$$

since the cross-product of a vector with itself is zero. Therefore, the derivative of the vector $\mathbf{r} \times \mathbf{v}$, which is $\mathbf{r} \times \dot{\mathbf{v}} + \mathbf{v} \times \mathbf{v}$, vanishes identically. Hence,

$$\mathbf{r} \times \mathbf{v} = \mathbf{c}, \tag{2.1}$$

where $\mathbf{c}$ is a constant vector. The vector $m\mathbf{c}$ is called the *moment of momentum* and its length mc the *angular momentum* of the particle. We ignore these refinements and refer to either $\mathbf{c}$ or c as the angular momentum. The assertion (2.1) is known as *the conservation of angular momentum.*

An important consequence of the principle can be deduced immediately. According to (2.1) we have $\mathbf{c} \cdot \mathbf{r} = 0$. If $c \neq 0$, this means that $\mathbf{r}$ is always perpendicular to the

fixed vector $\mathbf{c}$. Consequently, if $c \neq 0$, *all the motion takes place in a fixed plane through the origin perpendicular to* $\mathbf{c}$.

If $c = 0$, a little more subtlety is needed. Let $\mathbf{u}$ be a differentiable vector function of time and u its length. Since $u^2 = \mathbf{u} \cdot \mathbf{u}$, it follows that $u\dot{u} = \mathbf{u} \cdot \dot{\mathbf{u}}$. Therefore, if $u \neq 0$, we have

$$\frac{d}{dt}\frac{\mathbf{u}}{u} = \frac{u\dot{\mathbf{u}} - \dot{u}\mathbf{u}}{u^2}$$

$$= \frac{(\mathbf{u} \cdot \mathbf{u})\dot{\mathbf{u}} - (\mathbf{u} \cdot \dot{\mathbf{u}})\mathbf{u}}{u^3},$$

or

$$\frac{d}{dt}\frac{\mathbf{u}}{u} = \frac{(\mathbf{u} \times \dot{\mathbf{u}}) \times \mathbf{u}}{u^3}, \tag{2.2}$$

according to the vector formula

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

As an application of (2.2), let $\mathbf{u} = \mathbf{r}$. Then (2.2) becomes

$$\frac{d}{dt}\frac{\mathbf{r}}{r} = \frac{(\mathbf{r} \times \mathbf{v}) \times \mathbf{r}}{r^3} = \frac{\mathbf{c} \times \mathbf{r}}{r^3}, \tag{2.3}$$

by (2.1). Therefore, if $c = 0$, the vector $\mathbf{r}/r$ is a constant, and *the motion takes place along a fixed straight line through the origin.*

In case $c \neq 0$, another important consequence can be deduced from (2.1). Introduce into the plane of motion a polar coordinate system centered at O and forming a right-handed system with the vector $\mathbf{c}$. (See Fig. 1.) Then $\mathbf{r} = [r \cos \theta, r \sin \theta, 0]$ and $\mathbf{c} = [0, 0, c]$. A simple computation shows that (2.1) yields $r^2\dot{\theta} = c$. According to the calculus, the rate at which area is swept out by a radius

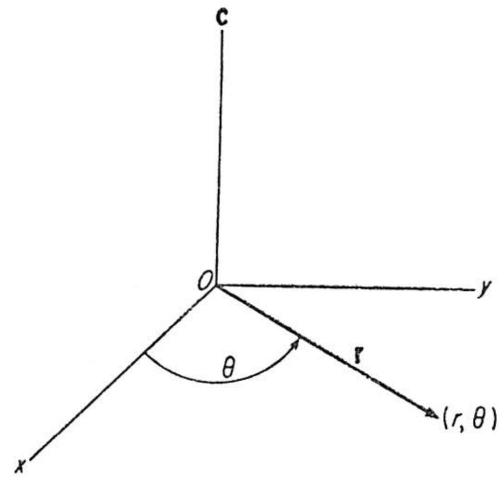

Figure 1

vector from O is just $\frac{1}{2}r^2\dot{\theta}$. Therefore *the particle sweeps out area at the constant rate* $c/2$. This fact is Kepler's *second law.*

EXERCISE 2.1. Set up the equations of motion of a particle moving subject to two distinct centers of attraction, each with its own law of attraction.

EXERCISE 2.2. Suppose that a particle subject to attraction by a fixed center starts from rest, i.e., that at some instant $t = 0$ we have $v = 0$. Then by (2.1) $c = 0$ and the motion is linear. Suppose, moreover, that $f(r)$ is positive for $0 < r < \infty$. Prove that the particle must collide with the center of force in a finite length of time t_0.

EXERCISE 2.3. In the preceding problem, can you tell where the particle will be at each instant of time

between 0 and t_0? First try the case $f(r) = \mu r^{-3}$ (inverse cube law), then $f(r) = \mu r^{-2}$ (inverse square law).

3. THE CONSERVATION OF ENERGY

So far we have found a vector **c** which remains constant throughout a particular motion. There is another constant of the motion which is of major importance, this time a scalar quantity called the *energy*. To find it, start with the second of Eqs. (1.1) and take the dot product of each side with **v**. We obtain

$$\begin{aligned} \dot{\mathbf{v}} \cdot \mathbf{v} &= -f(r)r^{-1}(\mathbf{r} \cdot \mathbf{v}) \\ &= -f(r)r^{-1}r\dot{r} \\ &= -f(r)\frac{dr}{dt}. \end{aligned}$$

Integration of both sides yields

$$\tfrac{1}{2}v^2 = f_1(r) + h, \tag{3.1}$$

where $f_1(r)$ is a function whose derivative is $-f(r)$ and h is a constant. The function $f_1(r)$ is determined conventionally this way:

$$f_1(r) = \int_r^a f(x)dx$$

where (i) a is chosen as $+\infty$ if the integral converges; (ii) a is chosen to be 0 if the first choice leads to a divergent integral but the second does not; (iii) a is chosen to be 1 if the first two choices fail. Thus, if $f(r)$ is of the form $f(r) = \mu r^{-p}$, then $a = \infty$ if $p > 1$; $a = 0$ if $p < 1$; $a = 1$ if $p = 1$. The most important case is that of Newton:

$$f(r) = \mu r^{-2}, \qquad f_1(r) = \mu r^{-1}.$$

With the above convention the function $-mf_1(r)$ is known as the *potential energy* and is denoted by the symbol $-U$. The quantity $T = mv^2/2$ is called the *kinetic energy*, and $h_1 = mh$ the *energy*. The statement (3.1) becomes

$$T = U + h_1, \tag{3.2}$$

and is known as the *principle of conservation of energy*.

EXERCISE 3.1. Show that if $f(r) = \mu r^{-p}$, where $p > 1$, then a particle moving with negative energy cannot move indefinitely far from O.

EXERCISE 3.2. Show that if $f(r) = \mu r^{-p}$, then $f_1(r) = \mu(p-1)^{-1}r^{1-p}$ if $p \neq 1$ and $f_1(r) = \mu \log 1/r$ if $p = 1$.

*EXERCISE 3.3. Let $\mathbf{a} = \mathbf{r}$, $\mathbf{b} = \mathbf{v}$ in the standard vector formula

$$(\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{a}\times\mathbf{b})^2 = a^2b^2.$$

Conclude that

$$v^2 = \dot{r}^2 + c^2r^{-2}.$$

What is the physical meaning of the components $\dot{r}$ and $c/\mathbf{r}$ of $\mathbf{v}$? Show that the law of conservation of energy can be written

$$r^2\dot{r}^2 + c^2 = 2r^2[f_1(r) + h].$$

4. THE INVERSE SQUARE LAW: KEPLER'S FIRST LAW

In this section we shall assume that the particle is moving according to Newton's law of gravitation. The governing

equations are then (1.2), which we repeat here for convenience as

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = -\mu r^{-3}\mathbf{r}. \tag{4.1}$$

It turns out that, in addition to the vector $\mathbf{c}$, there is another important vector which remains constant throughout the motion. It does not have a name in astronomical literature. We shall call it the *eccentric axis* and denote it by the symbol $\mathbf{e}$. To derive it, start with the formula (2.3) and multiply both sides by $-\mu$. Then

$$-\mu\frac{d}{dt}\frac{\mathbf{r}}{r} = \mathbf{c}\times(-\mu r^{-3}\mathbf{r}).$$

According to the second of Eqs. (4.1), this becomes

$$\mu\frac{d}{dt}\frac{\mathbf{r}}{r} = \dot{\mathbf{v}}\times\mathbf{c}.$$

Integration of both sides yields

$$\mu\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right) = \mathbf{v}\times\mathbf{c}, \tag{4.2}$$

where $\mathbf{e}$ is a constant of integration.

Since $\mathbf{r}\cdot\mathbf{c} = 0$, it follows that $\mathbf{e}\cdot\mathbf{c} = 0$. Hence, if $c \neq 0$, the vectors $\mathbf{e}$ and $\mathbf{c}$ are perpendicular, so that $\mathbf{e}$ lies in the plane of motion. If $c = 0$, $\mathbf{r}/r = -\mathbf{e}$, so that $\mathbf{e}$ lies along the line of motion; in this case the length e of $\mathbf{e}$ is always 1.

We shall now find the interpretation of e when $c \neq 0$. Take the dot product of both sides of (4.2) with $\mathbf{r}$. Then

$$\mu(\mathbf{e}\cdot\mathbf{r} + r) = \mathbf{r}\cdot\mathbf{v}\times\mathbf{c} = \mathbf{r}\times\mathbf{v}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{c}.$$

Consequently,

$$\mathbf{e}\cdot\mathbf{r} + r = c^2/\mu. \tag{4.3}$$

There are two cases. If $e = 0$, then $r = c^2/\mu$, a constant. Therefore the motion is circular. Moreover, according to the formula $r^2v^2 = r^2\dot{r}^2 + c^2$ of Ex. 3.3, it follows that $rv = c$, $v = \mu/c$, so that the particle moves with constant speed. By the law of conservation of energy, $v^2/2 = \mu/r + h$. Therefore $h = -\mu^2/2c^2$, a negative number. Observe finally that $2T = U$.

Suppose now that $e \neq 0$. In the plane of motion indicated by Fig. 1, introduce the vector **e** as shown in Fig. 2. The fixed angle from the x-axis to **e** will be denoted by ω. If (r, θ) represents a position Q of the particle, the angle $\theta - \omega$ will be denoted by f. The same position of the particle can then be represented as (r, f) if **e** is used as the axis of coordinates. It follows that $\mathbf{e} \cdot \mathbf{r} = er \cos f$ and Eq. (4.3) becomes

$$r = \frac{c^2/\mu}{1 + e \cos f}\,. \tag{4.4}$$

Consider the dotted line L in Fig. 2 drawn at a distance $c^2/\mu e$ from O, perpendicular to **e** and on the side of O to which **e** is directed. Equation (4.4), which can also be written $r = e\left(\dfrac{c^2}{\mu e} - r \cos f\right)$, simply says that the distance of the particle at Q from O is e times its distance from L. Consequently, *the particle moves on a conic section of eccentricity e with one focus at O*. This is Kepler's *first law*.

As (4.4) shows, the value of r is smallest when $f = 0$, since $e > 0$. Therefore the vector **e** is of length equal to the eccentricity and points to the position P at which the particle is closest to the focus.

There is some traditional terminology used by the astronomers that the reader ought to know. The position

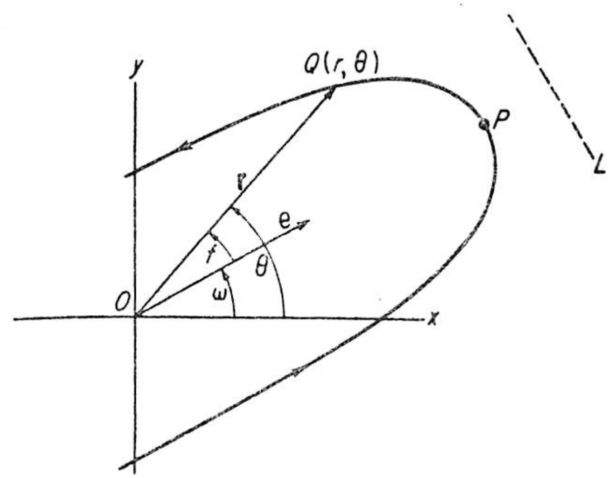

Figure 2

P is called the *pericenter*, the angle f the *true anomaly*. Various names are given to the pericenter, according to the source of attraction at O. If the source is the sun, P is called *perihelion*; if the earth, *perigee*; if a star, *periastron*. In the study of the solar system, the x-axis of Fig. 1 is fixed by astronomical convention. In that case, ω is the *argument of pericenter*.

We return to the geometry. The word *orbit* will be used to describe the set of positions occupied by the particle without any indication of the time at which a particular position is occupied. From the theory of conics it follows that if $0 < e < 1$ the orbit falls on an ellipse; if $e = 1$, on a parabola; and if $e > 1$, on a branch of hyperbola convex to the focus. Remember that in each case $c > 0$.

Since $r^2\dot{\theta} = c$ and $\dot{\theta} = \dot{f}$, it follows that $\dot{f} > 0$, so that the orbit is traced out in the direction of increasing f. This is indicated by the arrows on the curve in Fig. 2.

*EXERCISE 4.1. Show that if $0 < e < 1$ or $e > 1$ the semi-major axis of the corresponding conic has length a given by the formula

$$\mu a|e^2 - 1| = c^2.$$

EXERCISE 4.2. Use (4.2) to obtain the formula

$$\mu\mathbf{e} = \left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\mathbf{v}.$$

5. RELATIONS AMONG THE CONSTANTS

We pause at this point to remind the reader of some basic facts about differential equations. Let $f_i(z_1, \ldots, z_n)$, $i = 1, \ldots, n$ represent n functions with continuous first partial derivatives in some region of n-dimensional space, and let $(\zeta_1, \ldots, \zeta_n)$ be a particular point of this region. Then the system of differential equations

$$\dot{z}_i = f_i(z_1, \ldots, z_n), \qquad i = 1, \ldots, n \tag{5.1}$$

will have a unique solution $z_i(t)$ defined in a neighborhood of $t = 0$, such that $z_i(0) = \zeta_i$, $i = 1, \ldots, n$.

Now consider the basic Eqs. (1.1) with the additional assumption that f has a continuous derivative. This includes the special cases $f(r) = \mu r^{-p}$. Each of the two Eqs. (1.1) stands in place of three scalar equations, so that the pair constitutes a system of order six of the form (5.1). Specifically, let x, y, z denote the components of $\mathbf{r}$ in a rectangular coordinate system and let α, β, γ denote the

components of $\mathbf{v}$. The equations become

$$\dot{x} = \alpha$$
$$\dot{y} = \beta$$
$$\dot{z} = \gamma$$
$$\dot{\alpha} = -f(r)r^{-1}x$$
$$\dot{\beta} = -f(r)r^{-1}y$$
$$\dot{\gamma} = -f(r)r^{-1}z,$$

where $r^2 = x^2 + y^2 + z^2$. It follows that there is a unique solution satisfying six prescribed values of $x, y, z, \alpha, \beta, \gamma$ at $t = 0$. In vector form this says that the system (1.1) has a unique solution $\mathbf{r}(t)$, $\mathbf{v}(t)$ taking on prescribed values $\mathbf{r}_0$, $\mathbf{v}_0$ at time $t = 0$. These values can be prescribed arbitrarily.

In the special case $f(r) = \mu r^{-2}$, we have found that each of the quantities $\mathbf{c}$, $\mathbf{e}$, h remains constant during the motion and is therefore determined by its value at $t = 0$:

$$\mathbf{c} = \mathbf{r}_0 \times \mathbf{v}_0,$$
$$\mathbf{e} = \mu^{-1}(\mathbf{v}_0 \times \mathbf{c}) - r_0^{-1}\mathbf{r}_0,$$
$$h = \tfrac{1}{2}v_0^2 - \mu r_0^{-1}.$$

Since $\mathbf{c}$, $\mathbf{e}$, h constitute *seven* scalar quantities, it follows that there must be relations among them. We have already seen that there is a relation between $\mathbf{c}$ and $\mathbf{e}$, namely $\mathbf{c}\cdot\mathbf{e} = 0$. Therefore at most six of the seven quantities can be independent. Actually there is still another relation among the seven which reduces the number to *five*; it will be seen later that no further reduction is possible.

To obtain the new relation, square both sides of Eq. (4.2). Since v is perpendicular to c, we can replace $(\mathbf{v} \times \mathbf{c})^2$ by v^2c^2 to obtain

$$\mu^2\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right)^2 = v^2c^2$$

or

$$\mu^2\left(e^2 + \frac{2}{r}\,\mathbf{e}\cdot\mathbf{r} + 1\right) = v^2c^2.$$

Replace v^2 by $2h + (2\mu/r)$ and $\mathbf{e}\cdot\mathbf{r}$ by $(c^2/\mu) - r$, according to Eq. (4.3). Then

$$\mu^2(e^2 - 1) = 2hc^2. \tag{5.2}$$

Notice that this agrees with the earlier results that $e = 1$ if $c = 0$ and $h = -\mu^2/2c$ if $e = 0$.

Equation (5.2) has the following important consequences. If $c \neq 0$, then $e < 1$, $e = 1$ or $e > 1$ according to whether the energy h is negative, zero, or positive. If $h \neq 0$ and $c \neq 0$ and a is the semi-major axis of the conic (see Ex. 4.1), then

$$a = \tfrac{1}{2}\,\mu|h|^{-1}. \tag{5.3}$$

From this and the energy formula $\frac{1}{2}v^2 = (\mu/r) + h$, we obtain these basic formulas:

$$\begin{aligned} v^2 &= \mu\left(\frac{2}{r} + \frac{1}{a}\right) && \text{if } h > 0;\\ v^2 &= \frac{2\mu}{r} && \text{if } h = 0;\\ v^2 &= \mu\left(\frac{2}{r} - \frac{1}{a}\right) && \text{if } h < 0. \end{aligned} \tag{5.4}$$

These formulas still hold if $c = 0$ provided we adopt (5.3) as the definition of a; we shall do so.

EXERCISE 5.1. What can you say about the orbit if $f(r) = -\mu r^{-2}$ rather than $f(r) = \mu r^{-2}$? This corresponds to a repulsive force rather than an attraction.

EXERCISE 5.2. Use (5.4) to prove that in the case of elliptical motion the speed of the particle at each position Q is the speed it would acquire in falling to Q from the circumference of a circle with center at O and radius equal to the major axis of the ellipse.

*EXERCISE 5.3. The area of an ellipse is $\pi a^2(1 - e^2)^{1/2}$. We already know that if $c \neq 0$ the particle sweeps out area at the rate $c/2$. Combine these facts to show that if $0 < e < 1$ the period p of a particle, that is, the time it takes to sweep out the area once, is given by the formula $p = (2\pi/\sqrt{\mu})a^{3/2}$. This is Kepler's *third law*.

*EXERCISE 5.4. Define the moment of inertia $2I$ by the formula $2I = mr^2$. Write $r^2 = (\mathbf{r}\cdot\mathbf{r})$ and prove that

$$\ddot{I} = 2T - U = T + h_1 = U + 2h_1.$$

In the case of circular motion I is constant so that $2T = U$, a result we already know from Sec. 4.

EXERCISE 5.5. (*Hard.*) Use the preceding exercise to prove that if $c \neq 0$, $h > 0$, then $r/|t|$ approaches $\sqrt{2h}$ as $|t| \to \infty$. (The hypothesis $c \neq 0$ rules out the possibility of a collision with the origin in a finite time.)

6. ORBITS UNDER NON-NEWTONIAN ATTRACTION

The elegant method used in Sec. 4 to obtain orbits is essentially due to Laplace (who, however, did not have the vector concept available to him). It is applicable specifically to Newton's law of attraction. In the general case another method must be used. We know that if $c = 0$ the orbit is linear, so we shall assume that $c \neq 0$. Moreover, we assume that $f(r)$ has a continuous derivative.

Let us first dispose of the case of circular motion $r = r_0$. By the principle of conservation of energy, v is also a constant v_0 so the motion is uniform. The normal acceleration in the plane of motion is v_0^2/r_0 and this must be balanced by the attraction $f(r_0)$. Therefore, $v_0^2 = r_0 f(r_0)$. Since the velocity vector is perpendicular to the radius vector, it follows from $\mathbf{r} \times \mathbf{v} = \mathbf{c}$ that $rv = c$. Hence, $r_0 v_0 = c$, so that $c^2 = r_0^3 f(r_0)$. On the other hand, according to Ex. 3.3, the law of conservation of energy can be written

$$r^2\dot{r}^2 + c^2 = 2r^2[f_1(r) + h]. \tag{6.1}$$

Since $\dot{r} = 0$, we conclude that $c^2 = 2r_0^2[f_1(r_0) + h]$. Therefore, circular motion implies the two relations

$$c^2 = r_0^3 f(r_0), \qquad c^2 = 2r_0^2[f_1(r_0) + h]. \tag{6.2}$$

Conversely, we shall show that if (6.2) holds for the value of r at some instant of time, say $t = 0$, then the particle moves uniformly in a circle of radius r_0. According to (6.1), the second of Eqs. (6.2) implies that $\dot{r}_0 = 0$.

We interrupt the argument at this point to obtain an important general formula. Starting with the equation $r^2 = \mathbf{r} \cdot \mathbf{r}$, we obtain $r\dot{r} = \mathbf{r} \cdot \mathbf{v}$ by differentiation. Another differentiation yields $r\ddot{r} + \dot{r}^2 = (\mathbf{r} \cdot \dot{\mathbf{v}}) + (\mathbf{v} \cdot \mathbf{v}) = (\mathbf{r} \cdot \dot{\mathbf{v}}) + v^2$. But (see Ex. 3.3) $v^2 = \dot{r}^2 + c^2r^{-2}$, so that $r\ddot{r} = (\mathbf{r} \cdot \dot{\mathbf{v}}) + c^2r^{-2}$. Since $\dot{\mathbf{v}} = -f(r)r^{-1}\mathbf{r}$, we have $(\mathbf{r} \cdot \dot{\mathbf{v}}) = -f(r)r^{-1}\mathbf{r} \cdot \mathbf{r} = -rf(r)$. Therefore $r\ddot{r} = -rf(r) + c^2r^{-2}$, or, finally,

$$\ddot{r} - c^2r^{-3} = -f(r). \tag{6.3}$$

We resume the argument. According to the first of Eqs. (6.2), Eq. (6.3) has the constant solution $r = r_0$. Moreover, since the values of r and $\dot{r}$ at $t = 0$ are given, the uniqueness theorem described in Sec. 5 tells us that this is the only possible solution. This completes the case of circular motion.

In the general case it is customary to start with (6.3) and remove the dependence on time by substitution from $r^2\dot{\theta} = c$. Specifically, let $r = \rho^{-1}$. Then $\dot{r} = -\rho^{-2}\dot{\rho} = -\rho^{-2}\rho'\dot{\theta} = -\rho^{-2}\rho' c r^{-2} = -c\rho'$, where the prime (′) denotes differentiation with respect to θ. Hence $\ddot{r} = -c\rho''\dot{\theta} = -c^2\rho''\rho^2$. Equation (6.3) becomes

$$\rho'' + \rho = c^{-2}\rho^{-2}f\left(\frac{1}{\rho}\right). \tag{6.4}$$

In general, this cannot be solved for ρ in terms of θ in any recognizable form and we content ourselves with some special cases.

Suppose first that $f(r) = \mu r^{-2}$, the Newtonian case. Then $\rho'' + \rho = \mu/c^2$. It follows that ρ has the form $(\mu/c^2) + A\cos\theta + B\sin\theta$ and its reciprocal r has the form demanded by (4.4), since $f = \theta - \omega$.

Another easy case is $f(r) = \mu r^{-3}$. Then $\rho'' + \rho = \mu c^{-2}\rho$ or $\rho'' + (1 - \mu c^{-2})\rho = 0$. The solutions of this are well known.

EXERCISE 6.1. Classify the solutions in the case $f(r) = \mu r^{-3}$ according to the sign of $1 - \mu c^{-2}$. What if $1 - \mu c^{-2} = 0$?