Fixing my error around

$$\lim_{n\to\infty} \underbrace{\sqrt{2+\sqrt{2+\sqrt{2}}}}_{n\text{-times}}$$

Set $a_1 = \sqrt{2}$
$a_2 = \sqrt{2+\sqrt{2}}$
$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$
$\vdots$

My big error was, at one point writing the recursion relation as $a_{n+1} = \sqrt{2+\sqrt{a_n}}$.
That's not it!

$$a_2 = \sqrt{2+\sqrt{2}} = \sqrt{2+a_1}$$
$$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} = \sqrt{2+a_2}$$
so.

$$\boxed{a_{n+1} = \sqrt{2+a_n}. \qquad (*)_n}$$

We correctly showed, by induction that

$$a_n < 2 \quad \text{for all } n.$$

Theorem. $\lim_{n\to\infty} a_n = 2$

call it "a_∞"

<u>Proof</u>: First assume the limit exists. Then, letting $n \to \infty$ in $(*)_n$ we get

$$a_\infty = \sqrt{2 + a_\infty}$$

where $a_\infty = \lim_{n\to\infty} a_n$.

Now simply note: $a_\infty = 2$ works!

$$2 = \sqrt{2+2} = \sqrt{4}$$

Or if you prefer, square both sides:

$$a_\infty^2 = 2 + a_\infty.$$

So a_∞ solves

$$x^2 - x - 2 = 0$$

But

$$x^2 - x - 2 = (x-2)(x+1)$$

& we know

$$a_\infty > 0.$$

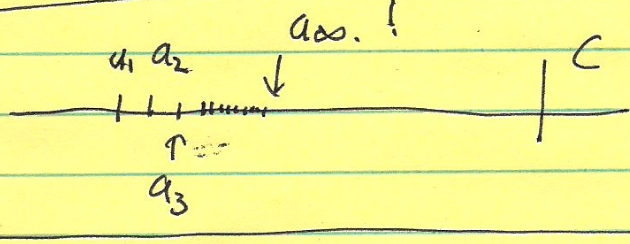
so...

$$a_\infty = 2 \qquad (\text{not } -1)$$

The criteria we were using for existence of limits is the following variant of the squeezing lemma:

Monotonicity ~~theor~~ lemma

$$\boxed{\begin{array}{l} \text{If a sequence of real numbers } a_n,\ n=1,2,3,\ldots \text{ is} \\ \quad \underline{\text{monotone}}: \quad a_n \le a_{n+1} \quad (\text{for all } n) \\ \& \ \underline{\text{bounded}} \quad a_n \le C \quad (\text{some } C \text{ indep. of } n) \\ \text{then } \lim_{n\to\infty} a_n \text{ exists} \\ \& \text{ is } \le C. \end{array}}$$



We (= me!) had failed to proved monotonicity.

How do we establish monotonicity (M)

$a_{n+1} \overset{?}{\geq} a_n$.

Use $a_{n+1} = \sqrt{2+a_n}$ & square:

$$a_{n+1}^2 = 2 + a_n.$$

So $a_{n+1} > a_n \iff a_{n+1}^2 > a_n^2$

$$\iff 2 + a_n > a_n^2$$

$$\iff 2 > a_n^2 - a_n.$$

Somehow, we are back to the function $x^2 - x - 2$. Let us look at the mapping:

$$x \longmapsto \sqrt{2+x} = F(x)$$

We computed its <u>only</u> fixed point:

$$F(2) = 2.$$

<u>Claim</u>: $F(x) > x$ for $0 < x < 2$.

<u>Note</u>: since we know each $a_n < 2$ ~~th~~ & since $a_{n+1} = F(a_n)$ this claim establishes monotonicity (M)

We now have a straight-up calculus / analytic geometry problem:

$$\boxed{\begin{array}{l}\underline{\text{Show}}\text{: for } 0 \le x < 2 \text{ that} \\ \qquad F(x) > x \\ \text{if } F(x) = \sqrt{2+x}\end{array}}$$

<u>Method of Solution</u>: Intermediate value theorem + algebra..

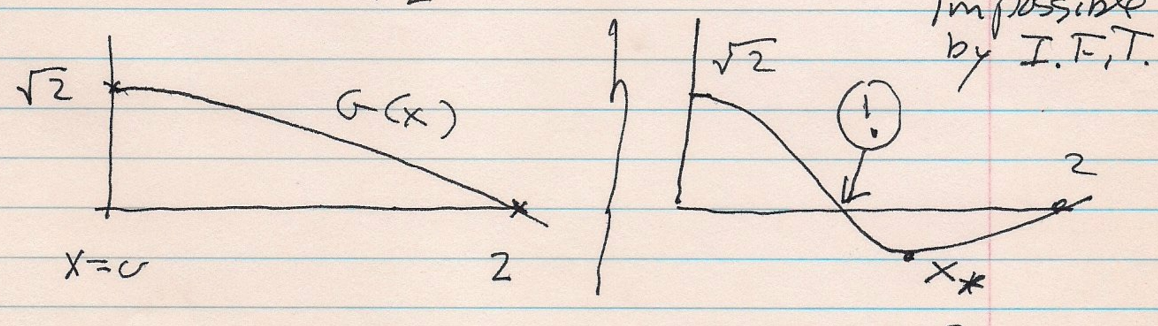
Set $G(x) = F(x) - x$.

By the algebra done above $G(x) = 0$

$G(x) = 0 \Leftarrow x = 2$ ← "the only zero of $G(x)$ is 2"

Now $G(x)$ is continuous, & $G(2) = 0$

Note $G(0) = \sqrt{2} - 0 > 0$.



Thus $G(x) > 0$ for all x, $0 \le x < 2$,

Because <u>if</u> there was an x_* with $G(x_*) < 0$, & $0 \le x_* < 2$, then by the IFT, $G(x)$ would have another zero.

Exercise

let $a_1 = 1$

$$a_2 = \sqrt{2+1}$$

$$a_3 = \sqrt{2+\sqrt{1+2}}$$

$$\cancel{a_3 = \sqrt{2+\sqrt{\cdot}}}$$

&

$$a_{n+1} = \sqrt{2+a_n}.$$

Show $\lim_{n\to\infty} a_n = 2.$

More generally, if we "seed" the map $x \longmapsto F(x) = \sqrt{2+x}$ with any number

$$a_0 = x_* \quad , \ 0 \leq x_* \leq 2.$$

& set $a_1 = F(a_0)$, $a_2 = F(a_1)$, ...

$$a_{n+1} = F(a_n)$$

then:

$$\lim_{n\to\infty} a_n = 2$$

$\uparrow$

Fixed point method.