Feb 28 $\underline{R}$ 1

Real numbers as decimals.

$$3.14159 \ldots \in \mathbb{R} =$$ set of all real numbers.

$$1.414 \ldots \in \mathbb{R}$$

$$.999.$$

Meaning of : , for π:

$$3 < \pi < 4$$

$$3\tfrac{1}{10} < \pi < 3\tfrac{2}{10}$$

$$3 + \frac{1}{10} + \frac{4}{100} < \pi < 3 + \frac{1}{10} + \frac{5}{100}$$

$$3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} < \pi < 3 + \frac{1}{10} + \frac{4}{100} + \frac{2}{1000}$$

etc.

Simpler,

$$.9999\ldots = 9 \times .1111\ldots$$

why $= 1$?

$$.111\cdots 1 = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

Example of a geometric series:

$$x + x^2 + x^3 + \cdots$$

or

$$1 + x + x^2 + \cdots$$

Feb 28 R2

Why is $.111\ldots = \frac{1}{9}$?

1st Formally: set

$$S = 1 + x + x^2 + x^3 + \cdots$$

then:

$$xS = x + x^2 + x^3 + \cdots$$

so

$$S - xS = 1 + \underbrace{x - x}_{0} + \underbrace{x^2 - x^2}_{0} - \cdots$$

$$= 1$$

So:

$$(1-x)S = 1$$

So

$$S = \frac{1}{1-x}.$$

~~Corollary~~

Theorem: If $-1 < x < 1$

then

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

&

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

Feb 28 R3

Examples

$.11111.. = x + x^2 + \cdots$ with $x = \frac{1}{10}$.

So: $.1111.. = \frac{1/10}{1-1/10} = \frac{10 \times 1/10}{10 \times (1-\frac{1}{10})}$

$= \frac{1}{9}$.

&

$.999\cdots 9 = 9 \times .111\cdots$

$= 9 \times \frac{1}{9} = 1$

Again. why is $.9999\cdots 9 \neq 1$??

(infinitesimals...)

Eg:

$.121212 \ldots = 12 \times .\underbrace{01}_{\frac{1}{100}}\underbrace{01}_{10^{-4}}010101$

$= 12 \times \left(\frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots\right)$

$= ??$

Feb 28 R3

1

Proof of Theorem on geometric serie.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = ?$$

Why?

$$.9999\dots = ?$$

Why?

Try: a finite geometric series

$$1 + x + x^2 + \cdots + x^n = S$$

then

$$\underline{x + x^2 + \cdots + x^n + x^{n+1} = xS}$$

so.

$$1 - x^{n+1} = S - xS = (1-x)S.$$

or.

$$S = \frac{1-x^{n+1}}{1-x} = \frac{x^{n+1}-1}{x-1}.$$

so:

$$1 + x + \cdots + x^{n} = \frac{1-x^{n+1}}{1-x}$$

$\boxed{\text{Can exactly sum!}}$

Feb 28_R4

So: For example: If $0<x<1$

&

$$S_n = 1+x+x^2+\cdots+x^n$$

we have:

$$S_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} < \frac{1}{1-x}$$

Claim: for $0 \le x < 1$

as $n \to \infty$ $\quad \frac{x^{n+1}}{1-x} \longrightarrow 0.$

Proof...?

case $x = \frac{1}{2}$.

Let $\varepsilon > 0$ be given.

Eventually $2^n > \frac{1}{\varepsilon}$

so $\frac{1}{2^n} < \varepsilon$

& $$\frac{x^{n+1}}{1-x} = \frac{\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = \frac{1}{2^n} < \varepsilon.$$

It's a fact: if $|x|<1$

then $x^n \longrightarrow 0$, So: $S_n \longrightarrow \frac{1}{1-x}$.

Feb 28 R5

Thus:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 2.$$

(Zeno's paradox).

or

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 1.$$

$$1+x+x^2+\cdots+x^n+\cdots = \frac{1}{1-x}$$

Plug in $x=2$.

$$1+2+2^2+\cdots+2^n+\cdots = \frac{1}{1-2} = -1$$

??

Uhoh?! Why fails?

R8

A real number ... α.

How do we describe it?
How do we find it?

<u>Way 1</u>. Cauchy sequences.

We list a sequence converging to it:

$\underbrace{1.414\ldots}_{s_n} \to \sqrt{2}$.

$s_0 = 1$, $s_1 = 1 + \frac{4}{10}$, $s_2 = 1 + \frac{4}{10} + \frac{1}{100}$ etc.

or: For golden mean:

$\underbrace{1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{1+1}}}_{s_n} \longrightarrow \frac{\sqrt{5}-1}{2}$.

$s_n = \frac{F_n}{F_{n+1}}$

"Cauchy sequences"

$|s_n - s_m| \to 0$.

R~~8~~7

"Dedekind cuts:"

Every subset of ~~the~~ rational numbers bounded from above has a least upper bound.

eg: $\{x : x \in \mathbb{Q} : x^2 < 2\}$.

least upper bound: $\sqrt{2}$.

or $0 < x < 1$

$$S = \left\{ \frac{1}{1-x} - \frac{x^{n+1}}{1-x} : n = 0, 1, 2, \ldots \right\}$$

least upper bound:

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$