Implicit Differentiation

Problem:

$$(x^2+y^2)^2 = 2(x^2-y^2)$$

describes a certain curve in the plane (the "Lemniscate of Bernoulli" or "figure eight").

¿What is the maximum value that y can achieve on this curve?

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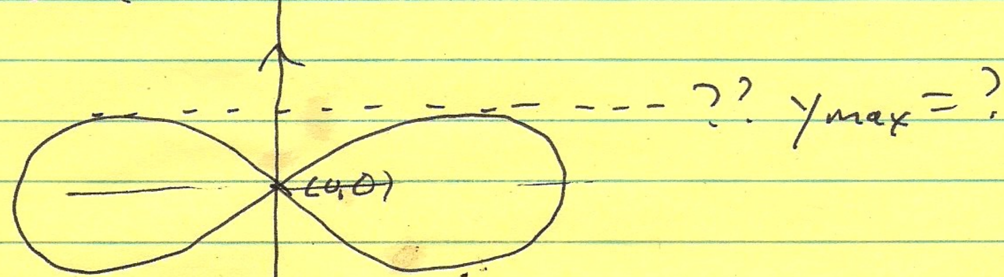
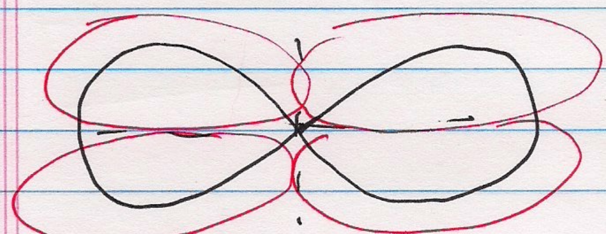
picture:



Solution: via Implicit differentiation.

$$(x^2+y^2)^2 = 2(x^2-y^2)$$



locally on curve.
4 regions where $y=y(x)$

Difficult to - impossible to solve:

ie. to find an explicit $y=y(x)$ such that

$$(x^2+y(x)^2)^2 = 2(x^2-y(x)^2)$$

But theory tells us, & our eye shows us, that this is true: curve looks like a graph away from origin & extremes on x-axis.

We can get information on $\frac{dy}{dx}$ without solving for $y=y(x)$

$\boxed{\text{Imagine}}$ we're solved.
Now differentiate:

$$\frac{d}{dx}\left(x^2+y(x)^2\right)^2 = 2(x^2+y^2)(2x+2yy')$$

$$= \frac{d}{dx} 2(x^2-y^2) = 2(2x-2yy').$$

or

$$4(x^2+y^2)yy' + 4(x^2+y^2)x$$

$$= 4x - 4yy'$$

so

$$4\left((x^2+y^2)+1\right) yy' = 4x - 4(x^2+y^2)x$$

so.

$$y' = \frac{x-(x^2+y^2)x}{\left((x^2+y^2)+1\right)y}$$

$$= \frac{x}{y}\left(\frac{1-(x^2+y^2)}{1+(x^2+y^2)}\right).$$

so, eg: $y'=0$ when $x^2+y^2=1$

$$1 = 2(x^2-y^2)$$

We know maxes occur when $y' = 0$.

from eqn: $y' = 0 \Leftrightarrow \underline{x = 0}$

or $\underbrace{1 - (x^2 + y^2) = 0}$

~~so~~

throw out $x = 0$: !

$(x^2 + y^2)^2 = -2y^2 \Rightarrow y = 0.$

~~then back to curve:~~

$\cancel{(x^2 + y^2)^2 = 2(x^2 - y^2)}$

$x^2 + y^2 = 1$

Use: $x^2 + y^2 = 1$.

Back in curve:

$(x^2 + y^2)^2 = 2(x^2 - y^2)$

or $1 = 2(x^2 - y^2)$.

So:

A) $x^2 + y^2 = 1$

B) $x^2 - y^2 = 1/2$.

Subtract B) from A): $2y^2 = 1/2$.

$$y^2 = \frac{1}{4}$$

$$y = \pm \frac{1}{2} \qquad ; y = \frac{1}{2} : \text{Max}$$

$$y = -\frac{1}{2} : \text{Min}$$

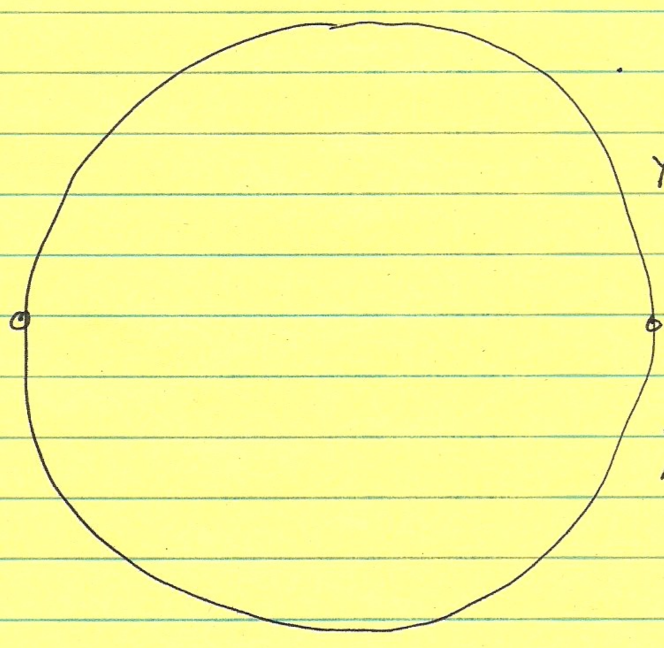
General theory,

applied to circle first.

$$x^2 + y^2 = 1.$$

Could solve

$y = y(x)$



$y = +\sqrt{}$

$y = -\sqrt{}$

Instead:

differentiate the eqn:

$$x^2 + y^2 = 1. \qquad (C)$$

$$\frac{d}{dx} \quad 2x + 2y\frac{dy}{dx} = 0 \quad \leftarrow$$

imagining: $y = y(x)$ so (C) holds.

Solving: $$y\frac{dy}{dx} = -x$$

or $$\frac{dy}{dx} = \frac{-x}{y}.$$

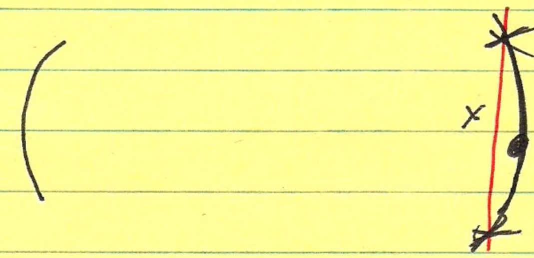
Verifying: $$y = \pm\sqrt{1-x^2}.$$

$$\frac{dy}{dx} = \pm\tfrac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \mp\frac{x}{\sqrt{1-x^2}} =$$

$$= -\frac{x}{y}.$$

What happens near $y=0$ where slope goes vertical?



No way to write $y=y(x)$ since 2 y-values for a given x-value

<u>Instead</u>: $x=x(y)$.

Again: $\frac{dx}{dy}$:

$$\frac{d}{dy}\left(x^2+y^2=1\right)$$

$$2x\frac{dy}{dx}+2y=0.$$

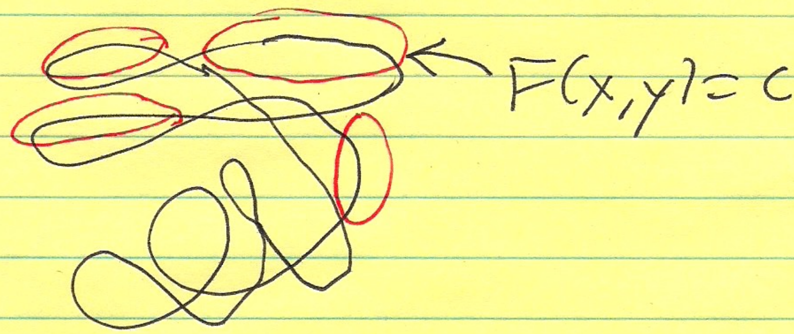
$$x\frac{dy}{dx}=-y; \quad \frac{dy}{dx}=\frac{-y}{x}$$

General scenario:

$F(x,y) =$ constant.

~~For nice 2F~~

defines a curve, perhaps quite complicated in the x-y plane:



On various pieces of the curve: $y = y(x)$

or

$x = x(y)$.

Meaning: on these regions of curve

$$F(\cancel{x}u, v) = c \iff v = y(u)$$

or

Suppose we want to understand how tangents change along curve, or, say, mins & maxes of y along curve - -

Compute:

$d(F = c)$

or

$dF = 0.$

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

total differential

pretend y a constant
compute $\frac{d}{dx}$

pretend x const.
compute $\frac{\partial}{dy}$

get:

$$\frac{\partial F}{\partial x}(x,y)\,dx + \frac{\partial F}{\partial y}(x,y)\,dy = 0.$$

Now depending on what function you want to be a function of what: divide:

eg: $y = y(x)$. $\Rightarrow$ $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

get

$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}.$$

requires: $\dfrac{\partial F}{\partial y} \neq 0$.

this is the condition of the <u>Implicit Function Theorem</u>

If the partial derivatives exist $\left(\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}\right)$ & are continuous ~~the~~ near a point (x_0, y_0) with

$$F(x_0, y_0) = c.$$

Then near~~th~~by to (x_0, y_0) there is a function

$$x \longmapsto \Psi(x) = y$$

with the significance that for these nearby (x, y)

$$F(x,y) = c \iff y = \Psi(x).$$

Back to circle.

$F = x^2 + y^2 = 1.$

$dF = 2x\,dx + 2y\,dy$

$= 0.$

$\Rightarrow \quad x\,dx + y\,dy = 0.$

$\Rightarrow \quad x + y\frac{dy}{dx} = 0$

or $\frac{dy}{dx} = -\frac{x}{y},$

good as long as $y \neq 0$.

ie.