5.9 Exponential Growth and Decay

In this section, we explore some applications of the expone... function. Conside... quantity $P(t)$ that depends exponentially on time:

$$P(t) = P_0 e^{kt}$$

The constant k has units of "inverse time"; if t is measured in days, then k has units of $(\text{days})^{-1}$.

If $k > 0$, then $P(t)$ *grows exponentially* and k is called the *growth constant*. Note tha... is the initial size (the size at $t = 0$):

$$P(0) = P_0 e^{k \cdot 0} = P_0$$

We can also write $P(t) = P_0 b^t$ with $b = e^k$, because $b^t = (e^k)^t = e^{kt}$.

A quantity that decreases exponentially is said to have *exponential decay*. In this... we write $P(t) = P_0 e^{-kt}$ with $k > 0$; k is then called the *decay constant*.

Population is a typical example of a quantity that grows exponentially, at least ... suitable conditions. To understand why, consider a cell colony with initial popul... $P_0 = 100$ and assume that each cell divides into two cells after 1 hour. Then popul... $P(t)$ doubles with each passing hour:

$$P(0) = 100 \quad \text{(initial population)}$$
$$P(1) = 2(100) = 200 \quad \text{(population doubles)}$$
$$P(2) = 2(200) = 400 \quad \text{(population doubles again)}$$

After t hours, $P(t) = (100)2^t$.

FIGURE 1 *E. coli* bacteria, found in the human intestine. (*Dr. Gary Gaugler/Science Source*)

Exponential growth cannot continue over long periods of time. A colony starting with one E. coli cell would grow to 5×10^{89} cells after 3 weeks—much more than the estimated number of atoms in the observable universe. In actual cell growth, the exponential phase is followed by a period in which growth slows and may decline.

EXAMPLE 1 In the laboratory, the number of *Escherichia coli* bacteria (Fig... grows exponentially with growth constant of $k = 0.41\ (\text{hours})^{-1}$. Assume that 100... teria are present at time $t = 0$.

(a) Find the formula for the number of bacteria $P(t)$ at time t.

(b) How large is the population after 5 hours (h)?

(c) When will the population reach 10,000?

Solution The growth is exponential, so $P(t) = P_0 e^{kt}$.

(a) The initial size is $P_0 = 1000$ and $k = 0.41$, so $P(t) = 1000e^{0.41t}$ (t in hours).

(b) After 5 h, $P(5) = 1000e^{0.41 \cdot 5} = 1000e^{2.05} \approx 7767.9$. Because the number of b... is a whole number, we round off the answer to 7768.

(c) The problem asks for the time t such that $P(t) = 10{,}000$, so we solve

$$1000e^{0.41t} = 10{,}000 \quad \Rightarrow \quad e^{0.41t} = \frac{10{,}000}{1000} = 10$$

Taking the logarithm of both sides, we obtain $\ln\left(e^{0.41t}\right) = \ln 10$, or

$$0.41t = \ln 10 \quad \Rightarrow \quad t = \frac{\ln 10}{0.41} \approx 5.62 \text{ h}$$

Therefore, $P(t)$ reaches 10,000 after approximately 5 h, 37 min (Figure 2).

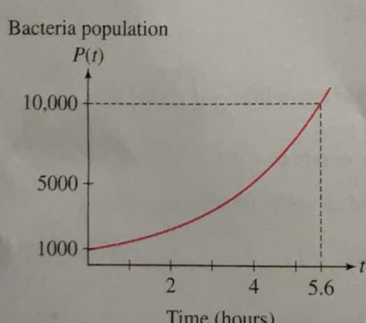


FIGURE 2 Growth of *E. coli* population.

The important role played by exponential functions is best understood in term... *differential equation* $y' = ky$. The function $y = P_0 e^{kt}$ satisfies this differential e... as we can check directly:

A differential equation is an equation relating a function $y = f(x)$ to its derivative y' (or higher derivatives y', y'', y''', ...).

$$y' = \frac{d}{dt}\left(P_0 e^{kt}\right) = kP_0 e^{kt} = ky$$

Theorem 1 goes further and asserts that the exponential functions are the *only* f... that satisfy this differential equation.

6791871

THEOREM 1 If y is a differentiable function satisfying the differential equation

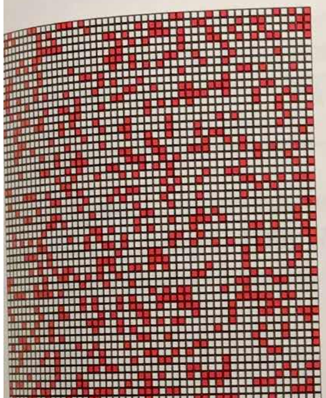
$$y' = ky$$

then $y(t) = P_0 e^{kt}$, where P_0 is the initial value $P_0 = y(0)$.

Proof Compute the derivative of ye^{-kt}. If $y' = ky$, then

$$\frac{d}{dt}(ye^{-kt}) = y'e^{-kt} - ke^{-kt}y = (ky)e^{-kt} - ke^{-kt}y = 0$$

Because the derivative is zero, $y(t)e^{-kt} = P_0$ for some constant P_0, and $y(t) = P_0 e^{kt}$ as claimed. The initial value is $y(0) = P_0 e^0 = P_0$. ■



URE 3 Computer simulation of dioactive decay as a random process. The d squares are atoms that have not yet cayed. A fixed fraction of red squares rns white in each unit of time. *(Courtesy of chael Zingale)*

CONCEPTUAL INSIGHT Theorem 1 tells us that a process obeys an exponential law precisely when *its rate of change is proportional to the amount present*. This helps us understand why certain quantities grow or decay exponentially.

A population grows exponentially because each organism contributes to growth through reproduction, and thus the growth rate is proportional to the population size. However, this is true only under certain conditions. If the organisms interact—say, by competing for food or mates—then the growth rate may not be proportional to population size and we cannot expect exponential growth.

Similarly, experiments show that radioactive substances decay exponentially. This suggests that radioactive decay is a random process in which a fixed fraction of atoms, randomly chosen, decays per unit time (Figure 3). If exponential decay were not observed, we might suspect that the decay was influenced by some interaction between the atoms.

■ **EXAMPLE 2** Find all solutions of $y' = 3y$. Which solution satisfies $y(0) = 9$?

Solution The solutions to $y' = 3y$ are the functions $y(t) = Ce^{3t}$, where C is the initial value $C = y(0)$. The particular solution satisfying $y(0) = 9$ is $y(t) = 9e^{3t}$. ■

■ **EXAMPLE 3 Modeling Penicillin** Pharmacologists have shown that penicillin leaves a person's bloodstream at a rate proportional to the amount present.

(a) Express this statement as a differential equation.

(b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours (h) after an initial injection of 450 mg.

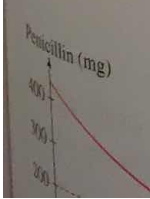
(c) Under the hypothesis of (b), at what time were 200 mg of penicillin present?

Solution

(a) Let $A(t)$ be the quantity of penicillin present in the bloodstream at time t. Since the rate at which penicillin leaves the bloodstream is proportional to $A(t)$,

$$A'(t) = -kA(t) \qquad (1)$$

where $k > 0$ because $A(t)$ is decreasing.



(b) Eq. (1) and the condition $A(0) = 450$ tell us that $A(t) = 450e^{-kt}$. The additional condition $A(7) = 50$ enables us to solve for k:

$$A(7) = 450e^{-7k} = 50 \quad \Rightarrow \quad e^{-7k} = \frac{1}{9} \quad \Rightarrow \quad -7k = \ln\frac{1}{9}$$

The constant k has units of time^{-1}, so the doubling time $T = (\ln 2)/k$ has units of time, as we should expect. A similar calculation shows that the tripling time is $(\ln 3)/k$, the quadrupling time is $(\ln 4)/k$, and, in general, the time to n-fold increase is $(\ln n)/k$.

There is a doubling ...th T. To prove this,

Quantities that grow exponentially possess an important pr... time T such that $P(t)$ doubles in size over every time interval ... let $P(t) = P_0 e^{kt}$ and solve for T in the equation $P(t+T) = 2$...

$$P_0 e^{k(t+T)} = 2P_0 e^{kt}$$

$$e^{kt} e^{kT} = 2e^{kt}$$

$$e^{kT} = 2$$

We obtain $kT = \ln 2$ or $T = (\ln 2)/k$.

Doubling Time If $P(t) = P_0 e^{kt}$ with $k > 0$, then the doubling time of P is

$$\text{Doubling time} = \frac{\ln 2}{k}$$

Number of hosts infected with Sapphire: 74855

FIGURE 5 Spread of the Sapphire computer virus 30 minutes after release. The infected hosts spewed billions of copies of the virus into cyberspace, significantly slowing Internet traffic and interfering with businesses, flight schedules, and automated teller machines. *(Copyright © 2010 The Regents of the University of California. All rights reserved. Used with permission.)*

■ **EXAMPLE 4 Spread of the Sapphire Worm** A computer virus nicknamed the *Sapphi... Worm* spread throughout the Internet on January 25, 2003 (Figure 5). Studies sugg... that during the first few minutes, the population of infected computer hosts increas... exponentially with growth constant $k = 0.0815\ \text{s}^{-1}$.

(a) What was the doubling time of the virus?

(b) If the virus began in four computers, how many hosts were infected after 2 minute... After 3 minutes (min)?

Solution

(a) The doubling time is $(\ln 2)/0.0815 \approx 8.5$ s (Figure 6).

(b) If $P_0 = 4$, the number of infected hosts after t seconds is $P(t) = 4e^{0.0815t}$. Afte... min (120 s), the number of infected hosts is

$$P(120) = 4e^{0.0815(120)} \approx 70{,}700$$

After 3 min, the number would have been $P(180) = 4e^{0.0815(180)} \approx 9.4$ million. Howe... it is estimated that a total of around 75,000 hosts were infected, so the exponential ph... of the virus could not have lasted much more than 2 min.

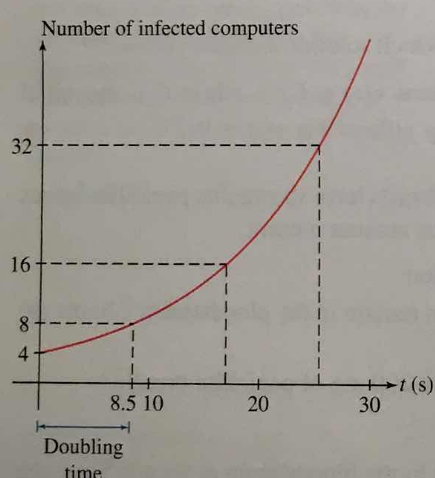


FIGURE 6 Doubling (from 4 to 8 to 16, etc.) occurs at equal time intervals.

In the situation of exponential decay $P(t) = P_0 e^{-kt}$, the **half-life** is the time it ta... for the quantity to decrease by a factor of $\frac{1}{2}$. The calculation similar to that of doubl... time above shows that

$$\text{Half-life} = \frac{\ln 2}{k}$$

■ **EXAMPLE 5** The isotope radon-222 decays exponentially with a half-life of 3... days. How long will it take for 80% of the isotope to decay?

Solution By the equation for half-life, k equals $\ln 2$ divided by half-life:

$$k = \frac{\ln 2}{3.825} \approx 0.181$$

Therefore, the quantity of radon-222 at time t is $R(t) = R_0 e^{-0.181t}$, where R_0 is... amount present at $t = 0$ (Figure 7). When 80% has decayed, 20% remains, so we s... for t in the equation $R_0 e^{-0.181t} = 0.2R_0$:

$$e^{-0.181t} = 0.2$$

$$-0.181t = \ln(0.2) \quad \Rightarrow \quad t = \frac{\ln(0.2)}{-0.181} \approx 8.9 \text{ days}$$

The quantity of radon-222 decreases by 80% after 8.9 days.

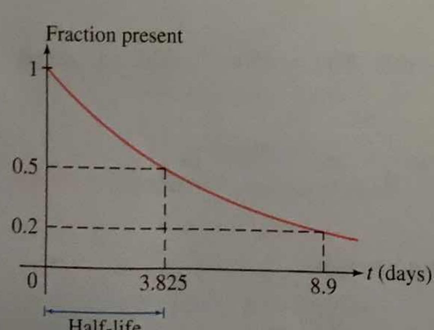


FIGURE 7 Fraction of radon-222 present at time t.

FIGURE 8 American chemist Willard Libby (1908–1980) developed the technique of carbon dating in 1946 to determine the age of fossils and was awarded the Nobel Prize in Chemistry for this work in 1960. Since then the technique has been refined considerably. (© Bettmann/Corbis)

Carbon Dating

Carbon dating (Figure 8) relies on the fact that all living organisms contain carbon that enters the food chain through the carbon dioxide absorbed by plants from the atmosphere. Carbon in the atmosphere is made up of nonradioactive C^{12} and a minute amount of radioactive C^{14} that decays into nitrogen. The ratio of C^{14} to C^{12} is approximately $R_{atm} = 10^{-12}$.

The carbon in a living organism has the same ratio R_{atm} because this carbon originates in the atmosphere, but when the organism dies, its carbon is no longer replenished. The C^{14} begins to decay exponentially while the C^{12} remains unchanged. Therefore, the ratio of C^{14} to C^{12} in the organism decreases exponentially. By measuring this ratio, we can determine when the death occurred. The decay constant for C^{14} is $k = 0.000121$ year^{-1}, so

$$\text{Ratio of } C^{14} \text{ to } C^{12} \text{ after } t \text{ years} = R_{atm}e^{-0.000121t}$$

EXAMPLE 6 Cave Paintings In 1940 a remarkable gallery of prehistoric animal paintings was discovered in the Lascaux cave in Dordogne, France (Figure 9). A charcoal sample from the cave walls had a C^{14}-to-C^{12} ratio equal to 15% of that found in the atmosphere. Approximately how old are the paintings?

Solution The C^{14}-to-C^{12} ratio in the charcoal is now equal to $0.15R_{atm}$, so

$$R_{atm}e^{-0.000121t} = 0.15R_{atm}$$

where t is the age of the paintings. We solve for t:

$$e^{-0.000121t} = 0.15$$

$$-0.000121t = \ln(0.15) \quad \Rightarrow \quad t = -\frac{\ln(0.15)}{0.000121} \approx 15{,}700$$

The cave paintings are approximately 16,000 years old (Figure 10). ■

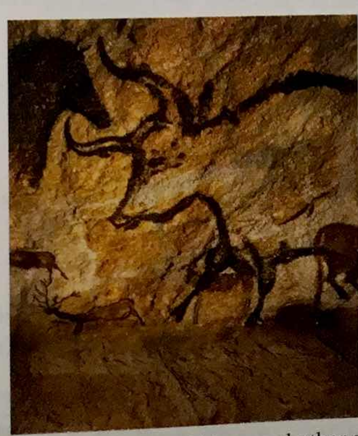
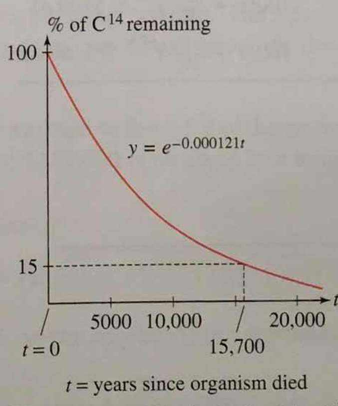


FIGURE 9 Detail of bison and other animals from a replica of the Lascaux cave mural.
(© Gianni Dagli Orti/Corbis)



DF FIGURE 10 If only 15% of the C^{14} remains, the object is approximately 16,000 years old.

Compound Interest and Present Value

Exponential functions are used extensively in financial calculations. Two basic applications are compound interest and present value.

When a sum of money P_0, called the **principal**, is deposited into an interest-bearing account, the amount or **balance** in the account at time t depends on two factors: the **interest rate** r and frequency with which interest is **compounded**. Interest paid out once a year at the end of the year is said to be *compounded annually*. The balance increases by the factor $(1 + r)$ after each year, leading to exponential growth:

Convention: Time t is measured in years and interest rates are given as yearly rates, either as a decimal or as a percentage. Thus, $r = 0.05$ corresponds to an interest rate of 5% per year.

	Principal	+	Interest		Balance
After 1 year	P_0	+	rP_0	=	$P_0(1+r)$
After 2 years	$P_0(1+r)$	+	$rP_0(1+r)$	=	$P_0(1+r)^2$
...		...			
After t years	$P_0(1+r)^{t-1}$	+	$rP_0(1+r)^{t-1}$	=	$P_0(1+r)^t$

Suppose that interest is paid out quarterly (every 3 months). Then the interest earned after 3 months is $\frac{r}{4}P_0$ dollars and the balance increases by the factor $\left(1+\frac{r}{4}\right)$. After 1 year (four quarters), the balance increases to $P_0\left(1+\frac{r}{4}\right)^4$ and after t years,

$$\text{Balance after } t \text{ years} = P_0\left(1+\frac{r}{4}\right)^{4t}$$

For example, if $P_0 = \$100$ and $r = 0.09$, then the balance after 1 year is

$$100\left(1+\frac{0.09}{4}\right)^4 = 100(1.0225)^4 \approx 100(1.09308) \approx \$109.31$$

More generally,

Compound Interest If P_0 dollars are deposited into an account earning interest at an annual rate r, compounded M times yearly, then the value of the account after t years is

$$P(t) = P_0\left(1+\frac{r}{M}\right)^{Mt}$$

The factor $\left(1+\frac{r}{M}\right)^M$ is called the **yearly multiplier**.

TABLE 1 Compound Interest with Principal $P_0 = \$100$ and $r = 0.09$

	Principal after 1 Year
Annual	$100(1+0.09) = \$109$
Quarterly	$100(1+\frac{0.09}{4})^4 \approx \$109.31$
Monthly	$100(1+\frac{0.09}{12})^{12} \approx \$109.38$
Weekly	$100(1+\frac{0.09}{52})^{52} \approx \$109.41$
Daily	$100(1+\frac{0.09}{365})^{365} \approx \$109.42$

Table 1 shows the effect of more frequent compounding. What happens in the limit as M tends to infinity? This question is answered by the next theorem (a proof is given at the end of this section).

THEOREM 2 Limit Formula for e and e^x

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{and} \quad e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \quad \text{for all } x$$

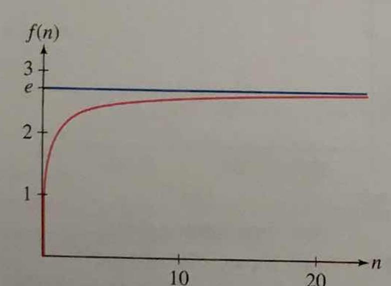


FIGURE 11 The function $f(n) = \left(1+\frac{1}{n}\right)^n$ approaches e as $n \to \infty$.

Figure 11 illustrates the first limit graphically. To compute the limit of the yearly multiplier as $M \to \infty$, we apply the second limit with $x = r$ and $n = M$:

$$\lim_{M\to\infty}\left(1+\frac{r}{M}\right)^M = e^r$$

The multiplier after t years is $(e^r)^t = e^{rt}$. This leads to the following definition.

Continuously Compounded Interest If P_0 dollars are deposited into an account earning interest at an annual rate r, compounded continuously, then the value of the account after t years is

$$P(t) = P_0e^{rt}$$

■ **EXAMPLE 7** A principal of $P_0 =$ ¥100,000 (Japanese yen) is deposited into an account paying 6% interest. Find the balance after 3 years if interest is compounded quarterly and if interest is compounded continuously.

Note: Th the sam pesos, y

In the fir different prime ra discussio one rate.

6791871

A second method of proof of Theorem 2 is to apply the methods of Section 4.5 to $\lim_{x\to\infty} f(x)^{g(x)}$, which, when we take $f(x) = 1 + \frac{1}{x}$ and $g(x) = x$, is an indeterminate form of type 1^∞.

5.9 SUMMARY

- *Exponential growth* with growth constant $k > 0$: $P(t) = P_0 e^{kt}$.
- *Exponential decay* with decay constant $k > 0$: $P(t) = P_0 e^{-kt}$.
- The solutions of the differential equation $y' = ky$ are the exponential functions $y = Ce^{kt}$, where C is a constant.
- A quantity $P(t)$ grows exponentially if it grows at a rate proportional to its size—that is, if $P'(t) = kP(t)$.
- The *doubling time* for exponential growth and the *half-life* for exponential decay are both equal to $(\ln 2)/k$.
- For use in carbon dating: The decay constant of C^{14} is $k = 0.000121$.
- Interest rate r, compounded M times per year:

$$P(t) = P_0(1 + r/M)^{Mt}$$

- Interest rate r, compounded continuously: $P(t) = P_0 e^{rt}$.
- The *present value* (PV) of P dollars (or other currency), to be paid t years in the future, is Pe^{-rt}.
- Present value of an income stream paying $R(t)$ dollars per year continuously for T years:

$$\text{PV} = \int_0^T R(t)e^{-rt}\,dt$$

ntially with growth constants
Which quantity doubles more

4. The PV of N dollars received at time T is (choose the correct answer):

(a) The value at time T of N dollars invested today

$y' = -0.7y$ and the initial condition $y(0) = 10$.

11. The decay constant of cobalt-60 is 0.13 year^{-1}. Find its half-life.

12. The half-life radium-226 is 1622 years. Find its decay constant.

13. One of the world's smallest flowering plants, *Wolffia globosa* (Figure 13), has a doubling time of approximately 30 hours (h). Find the growth constant k and determine the initial population if the population grew to 1000 after 48 h.

FIGURE 13 The tiny plants are *Wolffia*, with plant bodies smaller than the head of a pin. *(Gerald D. Carr)*

14. A 10-kg quantity of a radioactive isotope decays to 3 kg after 17 years. Find the decay constant of the isotope.

15. The population of a city is $P(t) = 2 \cdot e^{0.06t}$ (in millions), where t is measured in years. Calculate the time it takes for the population to double, to triple, and to increase 7 fold.

16. What is the differential equation satisfied by $P(t)$, the number of infected computer hosts in Example 4? Over which time interval would $P(t)$ increase 100 fold?

17. The decay constant for a certain drug is $k = 0.35\ \text{day}^{-1}$. Calculate the time it takes for the quantity present in the bloodstream to decrease by half, by one-third, and by one-tenth.

18. Light Intensity The intensity of light passing through an absorb-

mo
sol
sit
wh
lig
sor

23.
had
App

24.
spec
preh
Hele
year
the
C^{14}

25.