$$\cdots \frac{}{1-u^2} = \frac{1}{2}\left(\tanh^{-1}(0.36) - \tanh^{-1}(0.04)\right) \approx 0.1684 \quad \blacksquare$$

Excursion: A Leap of Imagination

The terms "hyperbolic sine" and "hyperbolic cosine" suggest a connection between the hyperbolic and trigonometric functions. This excursion explores the source of this connection, which leads us to **complex numbers** and a famous formula of Euler (Figure 1).

Recall that $y = e^t$ satisfies the differential equation $y' = y$. In fact, we know that *every* solution is of the form $y = Ce^t$ for some constant C. Observe that both $y = e^t$ and $y = e^{-t}$ satisfy the **second-order differential equation**

$$y'' = y \tag{2}$$

Indeed, $(e^t)'' = e^t$ and $(e^{-t})'' = (-e^{-t})' = e^{-t}$. Furthermore, *every* solution of Eq. (2) has the form $y = Ae^t + Be^{-t}$ for some constants A and B (Exercise 44).

Now let's see what happens when we change Eq. (2) by a minus sign:

$$y'' = -y \tag{3}$$

In this case, $y = \sin t$ and $y = \cos t$ are solutions because

$$(\sin t)'' = (\cos t)' = -\sin t, \qquad (\cos t)'' = (-\sin t)' = -\cos t$$

And as before, every solution of Eq. (3) has the form

$$y = A\cos t + B\sin t$$

This might seem to be the end of the story. However, we can also write down solutions of Eq. (3) using the exponential functions $y = e^{it}$ and $y = e^{-it}$. Here,

$$i = \sqrt{-1}$$

is an *imaginary* complex number satisfying $i^2 = -1$. Since i is not a real number, e^{it} is not defined without further explanation. But let's assume that e^{it} *can* be defined and that the usual rules of calculus apply:

$$(e^{it})' = ie^{it}$$

$$(e^{it})'' = (ie^{it})' = i^2e^{it} = -e^{it}$$

This shows that $y = e^{it}$ is a solution of $y'' = -y$, so there must exist constants A and B such that

$$e^{it} = A\cos t + B\sin t \tag{4}$$

The constants are determined by initial conditions. First, set $t = 0$ in Eq. (4):

$$1 = e^{i0} = A\cos 0 + B\sin 0 = A$$

Then take the derivative of Eq. (4) and set $t = 0$:

$$ie^{it} = \frac{d}{dt}e^{it} = A\cos' t + B\sin' t = -A\sin t + B\cos t$$

$$i = ie^{i0} = -A\sin 0 + B\cos 0 = B$$

Thus, $A = 1$ and $B = i$, and Eq. (4) yields **Euler's Formula**:

$$e^{it} = \cos t + i\sin t$$

Euler proved his formula using power series, which may be used to define e^{it} in a precise fashion. At $t = \pi$, Euler's Formula yields

$$e^{i\pi} = -1$$

Here, we have a simple but surprising relation among the four important numbers e, i, π, and -1.

Euler's Formula also reveals the source of the analogy between hyperbolic and trigonometric functions. Let us calculate the hyperbolic cosine at $x = it$:

$$\cosh(it) = \frac{e^{it} + e^{-it}}{2} = \frac{\cos t + i\sin t}{2} + \frac{\cos(-t) + i\sin(-t)}{2} = \cos t$$

A similar calculation shows that $\sinh(it) = i\sin t$. In other words, the hyperbolic and trigonometric functions are not merely analogous—once we introduce complex numbers, we see that they are very nearly the same functions.

7.4 SUMMARY