Formal minimization, maximization.

You have some function $f(x)$ of a real variable.

Maybe it only makes sense for x in some interval say:

$$-1 \le x \le 1.$$

& say f's derivatives all exist & even its 2nd derivatives

How do you find the minimizer(s)?

The maximum value, & max. point?

Example: Minimize $f(x) = x^3 - x$ over the interval

$$[-1, 1] = \{x : -1 \le x \le 1\}.$$

Tasks:

Compute: $f(+1)$ ← endpoint values

$f(-1)$

&

- Solve $f'(x) = 0$.

- check if any of these solutions lie in $[-1, 1]$.

$f(1) = ?$ $1^3 - 1$

$f(-1) = ?$

Note: $f(-x) = -f(x)$; $f(-1) = 0 = -f(1)$

$f'(x) = 3x^2 - 1$.

done! & quiz!

are values in $[-1, 1]$?

yes! $\sqrt{3} > 1$ so $-1 < \frac{-1}{\sqrt{3}} < 0 < \frac{1}{\sqrt{3}} < 1$.

$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{1}{\sqrt{3}}\right) \quad \lesseqgtr 0!$

$= \frac{1}{\sqrt{3}}\left(\left(\frac{1}{\sqrt{3}}\right)^2 - 1\right)$

$= \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{-2}{3\sqrt{3}}$.

Quiz 2'
answer

0. Problem.

Minimize $f(x) = x^3 - x$

on $-1 \le x \le 1$.

graph: $f(1) = 1^3 - 1 = 0$

$f(-1) = (-1)^3 - 1 = -2$.

$f(0) = 0$

From Quiz

$$f'(x) = 3x^2 - 1$$

$$f'(x) = 0 \iff 3x^2 = 1$$

$$x = \pm \frac{1}{\sqrt{3}}.$$

$\sqrt{3} > 1$ so $-1 \le \pm\frac{1}{\sqrt{3}} \le 1$.

$$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\left(\left(\frac{1}{\sqrt{3}}\right)^2 - 1\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = -\frac{2}{3\sqrt{3}}.$$

$$f\left(-\tfrac{1}{\sqrt{3}}\right) = -f\left(\tfrac{1}{\sqrt{3}}\right)$$

since f is odd.

so $f\left(-\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$.

Tabulate values

	Extreme points	Extreme values.
endpts.	-1	0
	1	0
crit. pts.	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}\frac{1}{\sqrt{3}}$
	$-\frac{1}{\sqrt{3}}$	$+\frac{2}{3}\frac{1}{\sqrt{3}}$.

~~Wha~~ One of these is the minimum value.

Which? ([illegible]), ~~occurs~~ when $x = -1$.

minimizes

¿ What if I changed the ~~problem~~ to

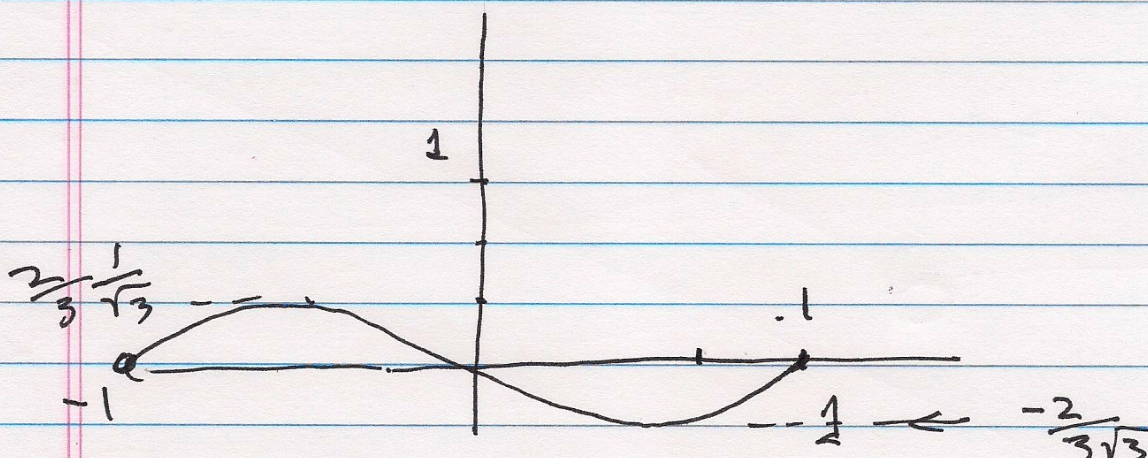
"maximize f"

over $[-1, 1]$.

(Value: $\frac{2}{3}\frac{1}{\sqrt{3}}$; Maximizer: $x = -\frac{1}{\sqrt{3}}$.

Min: occurs when $\boxed{x = \frac{1}{\sqrt{3}}} \simeq \frac{1}{1.7} \simeq .$

Min value $\boxed{-\frac{2}{3}\frac{1}{\sqrt{3}}} \simeq -\frac{2}{3}\frac{1}{1.7} \simeq -2/6.$



What if I ask you to minimize ~~f~~ the same $f(x)$ for $-1 \leq x \leq 0.$?

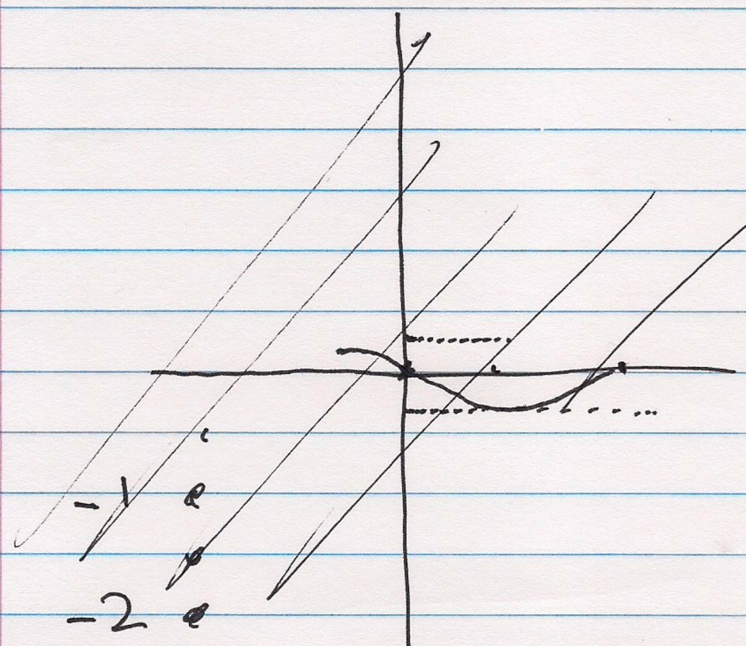
For $-2 \leq x \leq 1$?

4b

This algorithm works for the HWs of 4.2

$\sqrt{3} \simeq 1.7$

$\frac{2}{3\sqrt{3}} \simeq \frac{2}{3 \cdot 1.7} \simeq \frac{2}{6}$



Try for

$f(x) = 5 - 2x$

$0 \le x \le 4$

In class

some set

Definition If

$$f : \Omega \to \mathbb{R}$$

is a real valued function

then a minimizer for

if a point

$\sigma_0 \in \Omega$ s.t.

$\forall \sigma \in \Omega$

minimum value

$$f(\sigma_0) \le f(\sigma)$$

We call σ_0 a "minimizer"

Why Ω? why not $\mathbb{R}$?

Could be: ~~Cows~~ Set of Cows,

stocks

Cars in used car lot.

water malecules in a tea pot.

The value $f(\sigma_0)$ is called the minimum value

How to define "maximizer" & maximum value??

Our problems $\Omega \subseteq \mathbb{R}$.

But some Ω's better than others.

Eg ~~if~~ Consider $\Omega = \mathbb{R}$.
& $f : \Omega \to \mathbb{R}$
by $f(x) = e^x$.

.

.

.

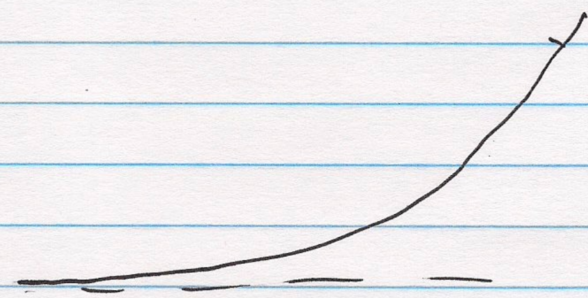
Example:

$$f(x) = e^x, \quad -\infty < x < \infty$$

$\cancel{-\infty}$

Min?

Max?



There is a limit or an "infimum"!

$$\text{"Min"} = 0 = \lim_{x \to -\infty} e^x$$

$$\text{"Max"} = +\infty = \lim_{x \to +\infty} e^x$$

But it is never reached.

Exer Find a nice

$f : (0,1) \to \mathbb{R}$

with no minimum.

Give ~~to~~ class. ; $(0,1) = \{x : 0 < x < 1\}$

!

The good Ω's.

closed bounded intervals:

$[a,b] = \{x : a \le x \le b\}$

Where $a, b \in \mathbb{R}$ are finite #s (Not ∞)

Thm: If Ω is a closed bounded interval & $f : \Omega \to \mathbb{R}$ is **continuous**

then f achieves its min. & max on Ω. ie. x_{min}, & x_{max} **exist**.

This Theorem is a liscence to find minimizes & ~~max's~~

Algorithm, when f's derivatives exist.

$\Omega = [a, b]$

boundary. mins

1). check values of $f(a)$, $f(b)$

interior mins.

2) compute $f'(x)$ & solve $f'(x) = 0$, $x \in \Omega$.

3) at all the values of 2, compute $f(x)$.

4) Compare all these values. find the minimum.

Variant.

Replace 3) with.

3'): At each of these x's, compute $f''(x)$ if $f''(x) < 0$ at one of these solutions throw it out: It is a local max.

Otherwise keep it.