Problem on Optimization
"Kepler's Wine Barrel Problem"
4.7. of Rog. edition 3, Problem #32.

A cylinder fits snugly inside a sphere. What ratio, height : diameter maximizes its volume, among all such cylinders?

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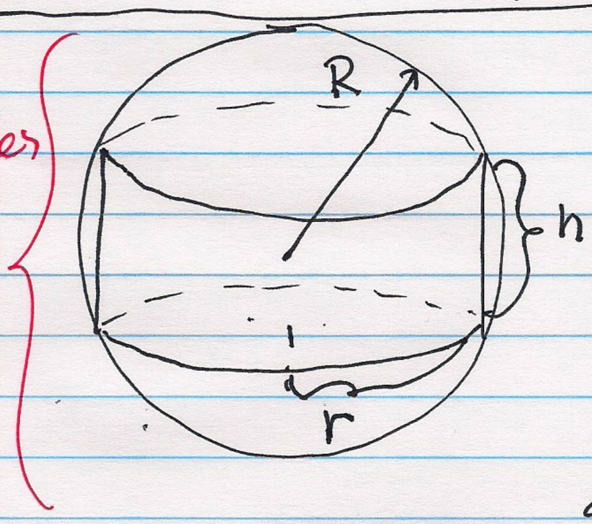
variables



let R be the sphere's radius, r the radius of the top & bottom circle of the cylinder & h the height of the cylinder.

By QUIZ (Pythagoras or Thales)

constraint

We have: $(2R)^2 = h^2 + (2r)^2$
or $R^2 = x^2 + r^2$

if $x = h/2$.

To maximize Volume
$$V = \pi r^2 h = 2\pi r^2 x$$

Method of ~~the~~ text

~~Alternative method, not using implicit differentiation:~~

Solve the constraint for r^2:

$$r^2 = R^2 - x^2$$

So:

$$V(x) = 2\pi \underbrace{(R^2 - x^2)}_{r^2} x$$

$$= 2\pi R^2 x - 2\pi x^3$$

&

$$\frac{dV}{dx} = 2\pi R^2 - 6\pi x^2$$

$$= 0.$$

$$\Rightarrow R^2 - 3x^2 = 0 \quad \text{or} \quad x^2 = \frac{1}{3}R^2$$

in which case: $r^2 = R^2 - x^2 = \frac{2}{3}R^2$

$$\cancel{\& \frac{h}{r} = \sqrt{\frac{(2x)^2}{r^2}} = \sqrt{\frac{4\cdot(\frac{1}{3}R^2)}{\frac{2}{3}R^2}}} = \sqrt{2}$$

✓.

Then $\dfrac{x^2}{r^2} = \dfrac{\frac{1}{3}R^2}{\frac{2}{3}R^2} = \dfrac{1}{2}$.

so $\dfrac{x}{r} = \sqrt{\dfrac{1}{2}} = \dfrac{1}{\sqrt{2}}$.

But height $= h = 2x$

diam $= D = 2r$.

$$\Rightarrow \boxed{\frac{h}{D} = \frac{x}{r} = \frac{1}{\sqrt{2}}}$$

Way 2. Use Implicit differentiation in

$$R^2 = x^2 + r^2$$

This eqn lets us think of either $r = r(x)$ or $x = x(r)$.

Choose one: $r = r(x)$. Then take $\frac{d}{dx}$ of both sides to get:

$$\text{(A)} \quad 0 = 2x + 2r(x)\frac{dr}{dx}.$$

Now differentiate V, using $r = r(x)$:

$$\text{(B)} \quad \frac{d}{dx}\left(2\pi r(x)^2 x\right) = 2\pi\left(2r\frac{dr}{dx}\right)x + 2\pi r^2$$

$$= 2\pi\left(2x\,r\frac{dr}{dx} + r^2\right)$$

But: from (A): $r\frac{dr}{dx} = -x$

so $\frac{dV}{dx} = 2\pi(-2x^2 + r^2)$

Setting $\frac{dV}{dx} = 0 \Rightarrow 2x^2 = r^2$

$$\Rightarrow \frac{x^2}{r^2} = \frac{1}{2} \Rightarrow \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{h}{\text{diam}}$$

since $h = 2x$, diam $= 2r$.