Intermediate Value Theorem

Bisection Algorithm.

Rolle's theorem &
Mean Value Theorem

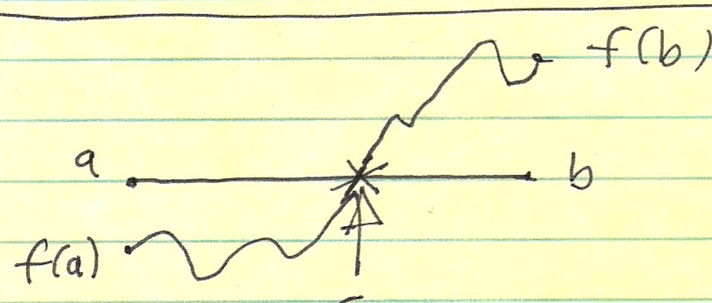
Continuity

Maybe: Newton iteration

Intermediate Value theorem

Say $a < b$
& f is continuous
& $f(a) < 0 < f(b)$
then! there is a number
c with $a < c < b$
& $f(c) = 0$.

Picture



Examp

Application

Root finding algorithm:

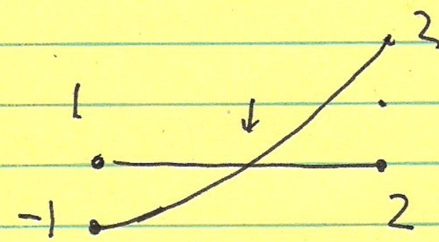
Bisection Method.

To solve

$$\boxed{x^2 = 2.}$$

Set

$$f(x) = x^2 - 2.$$



- f is continuous:
- a zero of f is a solution to $x^2 = 2$.
- $f(1) = -1$, $f(2) = 4 - 2 = 2$

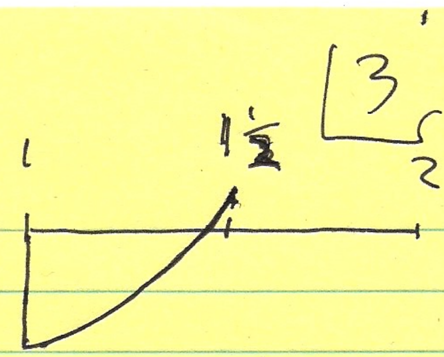
so between 1 & 2 there is a value c, $f(c) = 0$.

Where?

Split the difference.

$$f\left(\tfrac{3}{2}\right) = \frac{9}{4} - 2 = \frac{8}{4} + \frac{1}{4} - 2$$

$$= \frac{1}{4}$$



$f(0) < 0 < f(\frac{3}{2})$

So the zero c lies between 1 & 1.5.

Continue splitting the difference:

$f(1\frac{1}{4}) = f(\frac{5}{4}) = \frac{25}{16} - 2$

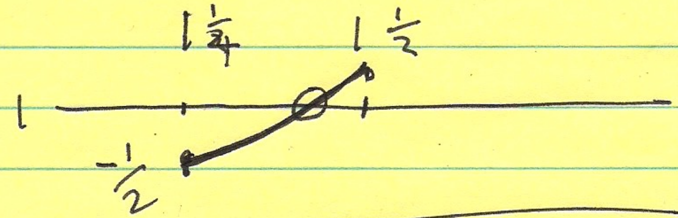
$\frac{32}{16} = 2$; $f(1\frac{1}{2}) < 0$.

$= -\frac{7}{16} \simeq -\frac{1}{2}$

So $1\frac{1}{4} < c < 1\frac{1}{2}$

or $1.25 < c < 1.5$

etc etc.



To Python Program

Theorems & Algorithms.

Continuity

Informal definitions:

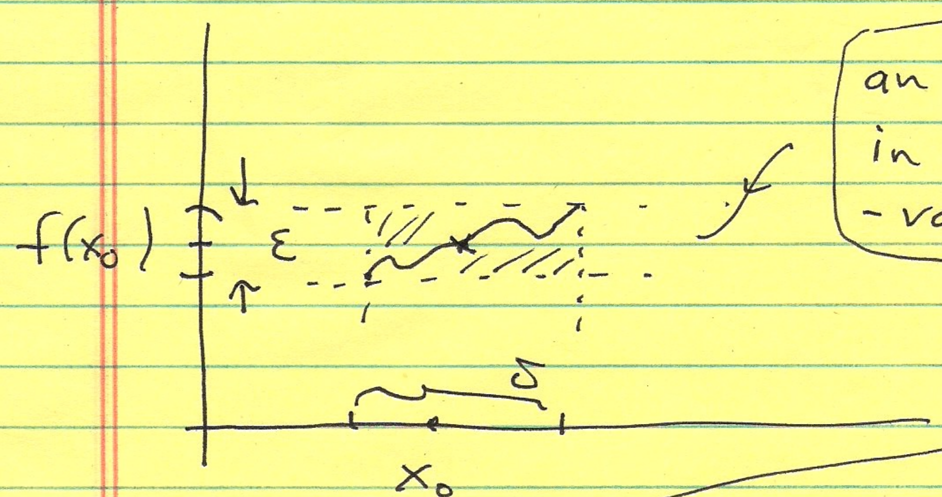
a) the function has no breaks

b) its graph has no sudden jumps.

Formal definition of Continuity

The function $y = f(x)$ is continuous at the point x_0 if, given **any** $\varepsilon > 0$ (no matter how small) there exists a $\delta = \delta(\varepsilon)$ (usually small) such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$



an ε-tolerance in the y-value

is guaranteed by a δ tolerance in the x value.

Definition: A real valued function defined on an interval or all of $\mathbb{R}$ is called continuous if it is continuous at all of its points.

Topological version of Intermediate value theorem.

If f is continuous on some interval J & $I = [a,b] \subset J$ is a closed bounded subinterval then $f(I) = \{f(x) : a \le x \le b\}$ is either a closed interval or a single point, (in which case f is constant on I).

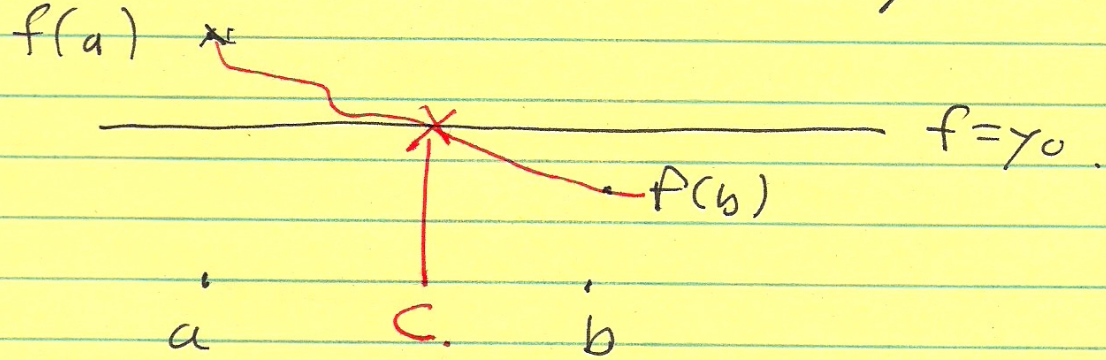
~~Eg~~ Cor: if $f(a) < f(b)$ then $f(I) \supset [f(a), f(b)]$.

__Cor__: If $f(a) < 0 < f(b)$ then $\exists\ c \in I$ $\underline{f(c) = 0}$

__Pf__: $0 \in [f(a), f(b)]$ in this case!

Usual intermediate value theorem

if y_0 lies between $f(a)$ & $f(b)$
then $\exists\, c$ between a & b & $f(c) = y_0$.



Mean Value Theorem

Oakland: ~~I~~ is 80 miles from here. You drove there in 1 hour.

Then: at some point you were going 80 mph (& so deserved a ticket)!

Math speak:

Thm

f continuous & differentiable on $[a,b]$. Suppose

$$v = \frac{f(b)-f(a)}{b-a}.$$

Then $\exists\ x_*$, with $a \le x_* < b$ & $f'(x_*) = v$

Do: WebAssign $\sqrt{x}$ $a=4$, $b=100$ problem. (of finding x_*)