Quiz: Compute the derivative of

$$f(x) = (x^7 + 4x^5 + x^3 + 3x^2 + 2x + 1)(x^5 + 3x^4 + 2x + 1)$$

in two different ways.

Recall from precalculus that there are many other functions aside from the polynomials we have so far focused on: e.g. trig, logarithmic, exponential, rational.

Today we will focus our attention on the exponential map $f(x) = e^x$. But before we can talk about its derivative we will need a good definition.

Define the exponential map as the function satisfying the initial value problem (a differential equation through a specified point)

$$y' = y \quad , \quad y(0) = 1.$$

Since our conversation has thus far focused on polynomials we will first try to see if maybe the above problem has a polynomial solution:

Suppose $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

then $y'(x) = \boxed{a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}}$

(hint: power rule)

~~How~~ Q: When can we say that two polynomials are equal?

A: When corresponding coefficients are equal.

Q: Is it possible then that $y(x)$ and $y'(x)$ as defined above are equal? Why?

A: No, by degree considerations or by noticing that $y(0) = a_0 = 1$.

(then $a_1 = 1$ also and $a_2 = a_3 = \cdots = a_n = 0$ so $y'(x) = 0$ which is contradictory)

The attempt seemed promising but the fact that degree is finite really tripped us up. So why not consider the "infinite degree" polynomial $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

Q: What does the ellipses (three dots) mean?

A: Continues in like manner.

As before, equate coefficients - but now we have $a_0 = 1$, $a_1 = 2a_2$, $a_2 = 3a_3$, $\ldots$; $a_{n-1} = n a_n$

so $a_2 = \frac{1}{2}$, $a_3 = \frac{a_2}{3} = \frac{1}{2 \cdot 3}$, $\ldots$

We generalize this (from $na_n = a_{n-1}$ ~~$na_{n+1} = a_n$~~)

$$a_n = \cancel{\frac{a_{n+1}}{a_n}} = \frac{a_{n-1}}{n}$$
$$= \frac{a_{n-2}/n-1}{n} = \frac{a_{n-2}}{n(n-1)}$$
$$= \frac{a_{n-3}/n-2}{n(n-1)} = \frac{a_{n-3}}{n(n-1)(n-2)}$$
$$\vdots$$
$$= \frac{a_0}{n\cdot(n-1)(n-2)\cdots 1} = \frac{1}{n!}$$

We have now that $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$ and, from our definition, $(e^x)' = e^x$.

To check that this is a good definition we ~~evaluate~~ approximate $y(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$

(used wolframalpha to compute the partial sum $y(1) \approx 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$ for $n = 2, 5, 10, 50$)

notice that at $n=10$ the approximation is so good that wolfram shows $y(1) - e \approx 0$.

Exercise: Compute the derivative of
$$f(x) = (x^4 + 3x + 1)(e^x + 2x + 1).$$

Now, why might we even consider the initial value problem used to define e^x?
Well, it "looks" a lot like the equation $\frac{d}{dt} P(t) = r P(t)$, which happens to be a decent simplification of population growth. Notice that as a population grows in magnitude, its rate of change does as well. This seems reasonable.

It turns out that this problem has general solution $P(t) = P(0) e^{rt}$.
(We'll need the chain rule to verify this solution but for now notice that if $r<0$, $P'(t)<0$ and $e^{rt}<1$ so long as $t>0$. This again is reasonable since if a population is decreasing, ~~it is~~ the current population is a fraction of the original. likewise if $r>0$, $P'(t)>0$ and $e^{rt}>1$ for $t>0$, which again seems reasonable.)

To understand the chain rule we will consider an example:

differentiate $(2x+1)^2$.

~~If we~~ If we first multiply and then differentiate (try this) or apply the product rule (try this) we have

$$((2x+1)^2)' = 2(2x+1) + (2x+1)(2)$$
$$= 2(2x+1)(2). \leftarrow \text{correct}$$

whereas if we (naively) apply just the product rule we get

$$((2x+1)^2)' = 2(2x+1) \leftarrow \text{in correct.}$$

We know the first case is correct since we just ~~did~~ used knowledge we are already comfortable with. The latter cannot be correct since it is different but notice it is off only by a constant multiple, $(2x+1)' = 2$.

We'll state the chain rule now, without further proof or example,

$$(f(g(x)))' = f'(g(x))\, g'(x).$$

~~To see~~

We now are able to verify the solution to the population problem from before.

If $P(t) = P(0)e^{rt}$

then $P'(t) = P(0)\,(e^{rt})\,(rt)'$

$= P(0)e^{rt}(r)$

$= r\,(P(0)e^{rt})$

$= r\,P(t).$