Final. Calculus (Math 19A). W 2018. [Montgomery, professor] 1 1 1

1. [10 pts] 10, RM.
2. [30] 30
3. [20] 20
4. [10] 10
5. [20] 20
6. [20] 20
7. [20] 20
8. [20] 20
9. [20] 20
10. [30] ~~25~~. 30

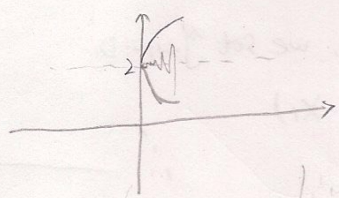
Σ: [200] ~~195~~ 200

For credit , show work.

RM

10/10 1. [10] If $f(x)$ is a function such that $2 - sin(2x) < f(x) < 2 + sin(2x)$ for $0 < x < \pi/2$ must the limit as $x \to 0$ of $f(x)$ exist? If so, what is the limit, and why?

According to squeeze lemma,
if $2 - \sin(2x) < f(x) < 2 + \sin(2x)$
for $0 < x < \frac{\pi}{2}$, because
$\lim_{x\to 0} 2 - \sin(2x) = \lim_{x\to 0} 2 + \sin(2x) = 2$
$\therefore \lim_{x\to 0} f(x) = 2$



Therefore, $\lim_{x\to 0} f(x)$ exists and its value equals 2.

2. [30] A) Write down the equation of the tangent line to the graph of the function $f(x) = x^3 - 25$ at the point (x, y) along the graph whose x-value is 3.

$f(x) = x^3 - 25$

$f'(x) = 3x^2$

When $x = 3$, $f(x) = 27 - 25 = 2$

$f'(x) = 3 \cdot 3 \cdot 3 = 27$

$\because y - y_0 = f'(x_0)(x - x_0)$

$\therefore y - 2 = 27(x - 3)$

$y = 27x - 81 + 2$

$= 27x - 79$

Therefore, the function of the tangent line is $y = 27x - 79$

B) Use your answer to part (A) to estimate the solution to the equation $x^3 = 25$. Express your answer as a rational number.

To find the solution of x, we need to know the x value when $f(x) = x^3 - 25 = 0$.
However, we cannot accurately determine the x value.
Therefore, we need the tangent line to estimate the x value.

We set the tangent line y value of the tangent line $y = 27x - 79$ to 0,

then $0 = 27x - 79$

$x = \frac{79}{27}$

C) Newton's algorithm leads to an iteration scheme $x_{i+1} = F(x_i)$ which when iterated converges to the zero of f. Find the explicit algebraic expression for this map F. **Sketch graphically [5 pts]** what F does to x_i. [Your answer to (B) should be $x_1 = F(x_0)$ if the 'seed' for the iteration is $x_0 = 3$.]

Using linear approximation, we know that $y_{i+1} - y_i = f'(x_i)(x_{i+1} - x_i)$

To approximate the solution, we set $y_{i+1} = 0$.

So $0 - y_i = f'(x_i)(x_{i+1} - x_i)$

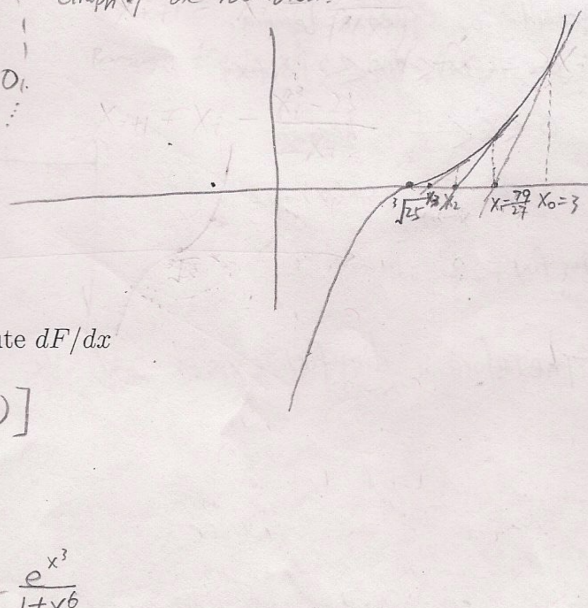
$-\frac{y_i}{f'(x_i)} = x_{i+1} - x_i$

$x_i - \frac{y_i}{f'(x_i)} = x_{i+1}$

Because $y_i = f(x_i)$, $x_i - \frac{f(x_i)}{f'(x_i)} = x_{i+1} \Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$.

Because $f(x_i) = x_i^3 - 25$ and $f'(x_i) = 2x_i$, $x_{i+1} = x_i - \frac{x_i^3 - 25}{3x_i^2}$

Graph of the iteration:



$x_0 = 3$

$x_1 = x_0 - \frac{x_0^3 - 25}{3x_0^2}$

$= 3 - \frac{27 - 25}{27}$

$= 3 - \frac{2}{27} = \frac{79}{27}$

$x_2 = \left(\frac{79}{27}\right) - \frac{\left(\frac{79}{27}\right)^3 - 25}{3 \cdot \left(\frac{79}{27}\right)^2}$

$x_3 = \cdots$

$\vdots$

x_i approaches to $\sqrt[3]{25}$.

3. [20] $F(x) = \int_1^{x^3} \frac{e^s}{1+s^2} ds$. Compute dF/dx

Assume $G'(s) = \frac{e^s}{1+s^2}$

$\frac{d}{dx} F(x) = \frac{d}{dx} \int_1^{x^3} \frac{e^s}{1+s^2} = \frac{d}{dx}\left[G(x^3) - G(1)\right]$

$= 3x^2 G'(x^3) - 0$

$= 3x^2 G'(x^3)$

Because $G'(s) = \frac{e^s}{1+s^2}$, $G'(x^3) = \frac{e^{x^3}}{1+(x^3)^2} = \frac{e^{x^3}}{1+x^6}$

$\therefore \frac{d}{dx} F(x) = 3x^2 G'(x^3) = \frac{3x^2 \cdot e^{x^3}}{1+x^6}$

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TA: N

4. [10] Evaluate $\lim_{x\to 0} \frac{1-x-e^{-x}}{x^2}$.

$\lim_{x\to 0} \frac{1-x-e^{-x}}{x^2} = \frac{1-0-e^0}{0} = \frac{1-0-1}{0} = \frac{0}{0} \Rightarrow$ indeterminate form.

So we apply L'Hopital's rule, $\lim_{x\to 0}\frac{1-x-e^{-x}}{x^2} = \lim_{x\to 0}\frac{(1-x-e^{-x})'}{(x^2)'} = \lim_{x\to 0}\frac{-1+e^{-x}}{2x} = \frac{-1+1}{0} = \frac{0}{0} \Rightarrow$ indeterminate form.

So we apply L'Hopital's rule again, and $\lim_{x\to 0}\frac{-1+e^{-x}}{2x} = \lim_{x\to 0}\frac{(-1+e^{-x})'}{(2x)'} = \lim_{x\to 0}\frac{-e^{-x}}{2} = \frac{-1}{2} = -\frac{1}{2}$

Therefore, $\lim_{x\to 0}\frac{1-x-e^{-x}}{x^2} = -\frac{1}{2}$

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5. [20] Compute the derivative $f'(x)$ when $f(x) = 3^{\cos(x)}/3^{\sin(x)}$.

$f(x) = \frac{3^{\cos(x)}}{3^{\sin(x)}} = 3^{\cos(x)-\sin(x)}$

$\therefore f'(x) = \frac{d}{dx}[3^{\cos(x)-\sin(x)}] = \ln(3)\cdot 3^{\cos(x)-\sin(x)} \cdot (\cos x - \sin x)'$

$= \ln(3)\cdot 3^{\cos(x)-\sin(x)} \cdot (-\sin x - \cos x)$

$= -\ln(3)\cdot 3^{\cos(x)-\sin(x)} \cdot (\sin x + \cos x)$

ln = loge.

6. [20] A certain polynomial $p(x)$ of degree ten satisfies $p(1) = -5$ and $p(3) = 17$. You also know that $p'(x) > 0$ for all x in the interval $[0, 5]$.

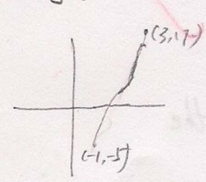
How many solutions are there to the equation $p(x) = 0$ on the interval $[1, 3]$? Why? [Hint : A correct answer to this last question will involve the name of some theorem covered in class.]

According to intermediate value theorem, if $f(x)$ is a continuous function in $[a,b]$ and $M \in [f(a), f(b)]$ there exists a value $c \in [a,b]$ such that $M = f(c) \in [f(a), f(b)]$

Here, a polynomial function is certainly continuous in $[1,3]$ and $0 \in [p(1), p(3)]$, then there exists a value $c \in [1,3]$ such that $p(c) = 0 \in [-5, 17]$

Therefore, $p(x)$ has at least one solution in the interval $[1,3]$.

Because $p'(x) > 0$ for all x in the interval $[0,5]$, $p(x)$ is monotonically increasing in $[0,5]$ and ~~thus~~ in $[1,3]$.



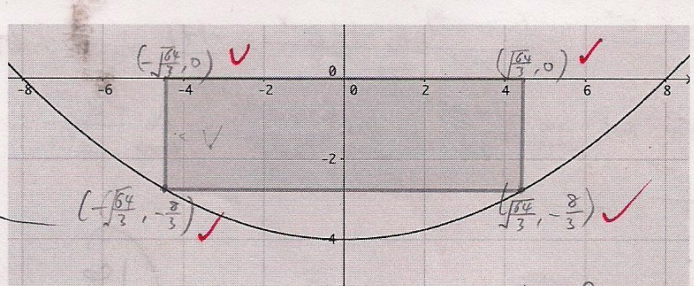
Therefore, there is only one c value such that $p(c) = 0$ in $[1,3]$

In other words, $p(c)$ only has one solution in $[1,3]$.

20

$\frac{\frac{64}{3}}{16} \cdot \frac{4}{3} - 4 = \frac{4-12}{3} = -\frac{8}{3}$

7. [20] Among all rectangles which lie below the x-axis and above the indicated parabola find the one containing the maximum area. Express your answer by finding the coordinates (x, y) of the four vertices of this maximizing rectangle.



$\sqrt{64} = 8$

$y = ax^2 + bx + c$

and $(-8, 0)$, $(8, 0)$, and $(0, -4)$ are on the parabola.

so: $\begin{cases} 64a - b + c = 0 \\ 64a + b + c = 0 \\ c = -4 \end{cases}$

$\Downarrow$

$\begin{cases} 64a - b = 4 \\ 64a + b = 4 \end{cases} \Rightarrow 2b = 0 \Rightarrow b = 0$

$\therefore 64a = 4$

$a = \frac{1}{16}$

$\therefore$ the function is $y = \frac{x^2}{16} - 4$

① Suppose x is positive and y is negative,

the area of rectangle

$= 2x(-y) = 2x \cdot (4 - \frac{x^2}{16})$

$= 8x - \frac{x^3}{8}$

Let $f(x) = 8x - \frac{x^3}{8}$

$f'(x) = 8 - \frac{3x^2}{8}$

when $f'(x) = 0$, $\frac{3x^2}{8} = 8$

$3x^2 = 64$

$x = \pm\frac{\sqrt{64}}{\sqrt{3}}$

Because $x > 0$, $x = \frac{\sqrt{64}}{\sqrt{3}}$, $y = -\frac{8}{3}$

$f''(x) = -\frac{6x}{8} = -\frac{3}{4}x$

$f''(\frac{\sqrt{64}}{\sqrt{3}}) < 0 \Rightarrow ∩ \Rightarrow$ max point

② Suppose x is negative and y is negative.

the area $= (-2x)(-y)$

$= -2x(4 - \frac{x^2}{16})$

$= \frac{x^3}{8} - 8x$

Let $g(x) = \frac{x^3}{8} - 8x$

$g'(x) = \frac{3x^2}{8} - 8$

when $g'(x) = 0$

$3x^2 = 64$

$x = \pm\sqrt{\frac{64}{3}}$

Because $x < 0$

$x = -\frac{\sqrt{64}}{\sqrt{3}}$ and $y = -\frac{8}{3}$

$f''(x) = \frac{6x}{8} = \frac{3}{4}x$

$f''(-\frac{\sqrt{64}}{\sqrt{3}}) < 0$ $∩ \Rightarrow$ max point for area

Therefore, the four points are $(-\frac{\sqrt{64}}{\sqrt{3}}, -\frac{8}{3})$, $(\frac{\sqrt{64}}{\sqrt{3}}, -\frac{8}{3})$, $(-\frac{\sqrt{64}}{\sqrt{3}}, 0)$, and $(\frac{\sqrt{64}}{\sqrt{3}}, 0)$.

8. [20] Use implicit differentiation to find the slope of the tangent line to the ellipse $x^2 + xy + y^2 = 1$ at the two points on the ellipse where $x = y$.

$x^2 + xy + y^2 = 1$

$(x^2 + xy + y^2)' = (1)'$

$2x + y + x \cdot y' + 2y \cdot y' = 0$

$(x + 2y)y' + 2x + y = 0$

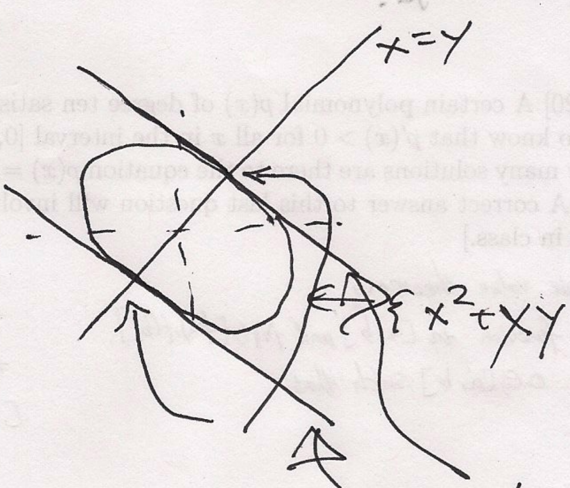
$y' = \frac{-2x - y}{x + 2y}$

when $x = y$

$y' = \frac{-2x - x}{x + 2x} = \frac{-3x}{3x} = -1$

Therefore, the slope of the tangent line is -1.

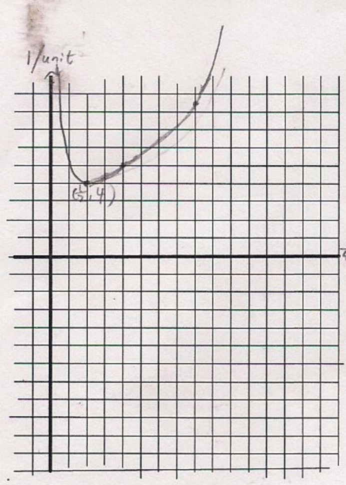
at either of the two points



these two tangent lines to the ellipse have slope -1

—RM

9. [20] Sketch the graph of $y = \frac{1}{x} + 4x$ for $x > 0$ on the graph paper below. Indicate the behaviour of the function as $x \to 0$ and $x \to \infty$. Indicate the precise location and value of the critical points. **Assume that the units on the paper are** $1/4 = .25$.



- $y = \frac{1}{x} + 4x$

$y' = -\frac{1}{x^2} + 4$

When $y' = 0$, $-\frac{1}{x^2} + 4 = 0$

$\frac{1}{x^2} = 4$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

Because $x > 0$, $x = \frac{1}{2} \Rightarrow y = 2 + 2 = 4$

- $y'' = \frac{2}{x^3}$

$y''|_{x=\frac{1}{2}} = 16 > 0 \Rightarrow \cup \Rightarrow$ the critical point is $(\frac{1}{2}, 4)$ and its the minimum point.

- When $x > 0$, $y'' > 0 \Rightarrow$ always concave up $\cup$.

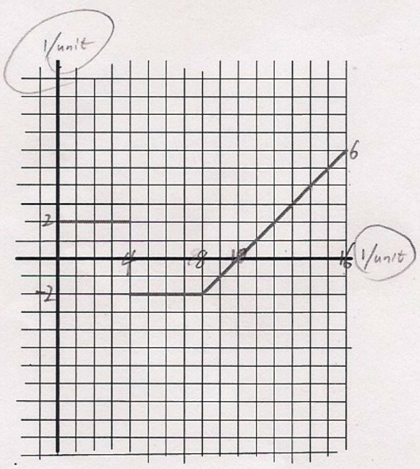
- $\lim_{x \to 0^+} \frac{1}{x} + 4x = \infty + 0 = +\infty$

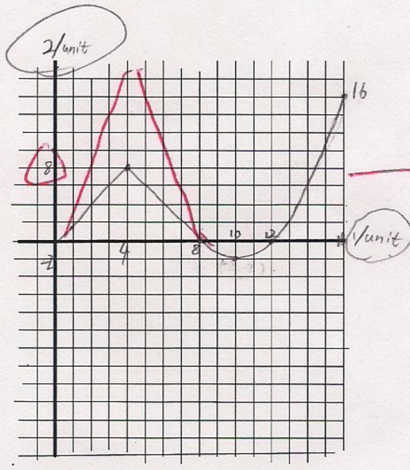
$\lim_{x \to \infty} \frac{1}{x} + 4x = 0 + \infty = \infty$

Therefore, the graph of the function can be drawn.

✓ 20

10. [30] The graph of a function $f(x)$ is given on the left. On the right is a blank graph. On both pieces of graph paper the thick black lines represent the coordinate axes so that $(0, 0)$ is their intersection. On the blank graph, sketch the graph of the indefinite integral function $F(x) = \int_0^x f(s)ds$ for $x > 0$.





~~25~~ oh!!

30

↑ y-units are 2x x units for him.