$[a_m, b_m]$ can be regarded as the m-th stage of the approximation. For the specific problem of Theorem 3.1, the above special property which characterizes an arbitrary small neighborhood of the target-place naturally provides the much needed guiding criterion for the inductive selection of subintervals, $\{[a_m, b_m],\ m \in \mathbb{N}\}$.

Exercises

To verify the existence of limit for the following sequences:

1. $a_1 = \sqrt{2}, a_2 = \sqrt{2+\sqrt{2}}, a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ and inductively $a_{n+1} = \sqrt{2+a_n}$.
2. $a_1 = \sqrt{2}, a_2 = \sqrt{2\cdot\sqrt{2}}, a_3 = \sqrt{2\cdot\sqrt{2\cdot\sqrt{2}}}, \ldots$ and inductively $a_{n+1} = \sqrt{2\cdot a_n}$.
3. $\left\{a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},\ n = 1,2,3,\ldots\right\}$.
4. $\left\{a_n = \left(1+\frac{1}{n}\right)^n,\ n = 1,2,3,\ldots\right\}$.
5. $\left\{b_n = \left(1+\frac{1}{n}\right)^{n+1},\ n = 1,2,3,\ldots\right\}$.
   [Hint: Compute $\frac{a_{n+1}}{a_n}$ and $\frac{b_n}{b_{n+1}}$ for Exercises 4 and 5 and make use of the inequality $\left(1-\frac{1}{(n+1)^2}\right)^{n+1} > 1-\frac{1}{n+1}$ to show that a_n (resp. b_n) are monotonically increasing (resp. decreasing).]
6. Suppose that $a_1 < b_1$ are two given positive real numbers. Set
$$a_2 = \sqrt{a_1 b_1},\ b_2 = \frac{1}{2}(a_1 + b_1)$$
$$a_3 = \sqrt{a_2 b_2},\ b_3 = \frac{1}{2}(a_2 + b_2)$$
   and inductively
$$a_{n+1} = \sqrt{a_n b_n},\ b_{n+1} = \frac{1}{2}(a_n + b_n).$$
   Prove that a_n (resp. b_n) are monotonically increasing (resp. decreasing), $a_n < b_n$ and $(b_n - a_n) \to 0$.

7. Let $0 < r < 1$. Show that $\lim_{n\to\infty} r^n = 0$.
8. Set $a_n = 1 + r + \cdots + r^{n-1}$. Show that

$$\lim_{n\to\infty} a_n = \frac{1}{1-r}$$

provided $|r| < 1$.
9. Show that every rational number can always be expressed as a cyclic decimal.
10. Show that the limit value of a cyclic decimal is always equal to a rational number.
[Thus irrational numbers are exactly those infinite decimals which are *non-cyclic*!]

§ 2. Limit and the Continuity of a Function

Geometrically speaking, a function $y = f(x)$ is continuous if its graph is a continuous curve. Intuitively, it roughly means that the change in the functional values will be small provided the change in the independent variable x is sufficiently small.

A mathematical definition of the localized concept of the continuity of $f(x)$ at $x = a$ was already given in Sec. 2 of Chapter 2. It is convenient to translate that definition in terms of sequential limit, namely,

Definition. A function $f(x)$ is said to be continuous at $x = x_0$ if to any given sequence $\{s_n\}$, $s_n \to x_0$, the corresponding sequence $\{f(s_n)\}$ converges to $f(x_0)$ as its limit, i.e., $f(s_n) \to f(x_0)$.

A function $f(x)$, $a \le x \le b$, is said to be continuous over $[a, b]$ if it is continuous at every point $x_0 \in [a, b]$.

Continuous functions over a closed interval forms an important family of functions in the study of calculus. They enjoy some nice properties which, in fact, plays a basic role in the framework of