Example 6. Sometimes the work can be shortened by a change of variable. For example could apply L'Hôpital's rule directly to calculate the limit

$$\lim_{x\to 0+} \frac{\sqrt{x}}{1 - e^{2\sqrt{x}}},$$

but we may avoid differentiation of square roots by writing $t = \sqrt{x}$ and noting that

$$\lim_{x\to 0+} \frac{\sqrt{x}}{1 - e^{2\sqrt{x}}} = \lim_{t\to 0+} \frac{t}{1 - e^{2t}} = \lim_{t\to 0+} \frac{1}{-2e^{2t}} = -\frac{1}{2}.$$

We turn now to the proof of Theorem 8–7.

Proof. We make use of Cauchy's mean-value formula (Theorem 7–4 of Section applied to a closed interval having a as its left endpoint. Since the functions f and g not be defined at a, we introduce two new functions that *are* defined there. Let

$$\text{(8.18)} \qquad \begin{aligned} F(x) &= f(x) \quad \text{if} \quad x \neq a, \qquad F(a) = 0, \\ G(x) &= g(x) \quad \text{if} \quad x \neq a, \qquad G(a) = 0. \end{aligned}$$

Because of (8.13), both F and G are continuous at a. In fact, if $a < x < b$, both funct F and G are continuous on the *closed interval* $[a, x]$ and have derivatives everywher the *open interval* (a, x). Therefore Cauchy's formula is applicable to the interval [and we obtain

$$[F(x) - F(a)]\, G'(c) = [G(x) - G(a)]\, F'(c),$$

where c is some point satisfying $a < c < x$. If we use (8.18), this becomes

$$\text{(8.19)} \qquad f(x)g'(c) = g(x)f'(c).$$

Now $g'(c) \neq 0$ [since, by hypothesis, g' is never zero in (a, b)] and also $g(x) \neq 0$ fact, if we had $g(x) = 0$ then we would have $G(x) = G(a) = 0$ and, by Rolle's theor there would be a point x_1 between a and x where $G'(x_1) = 0$, contradicting the hypot that g' is never zero in (a, b). Therefore we may divide by $g'(c)$ and $g(x)$ in (8.19) to ob

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

As $x \to a$, the point $c \to a$ (since $a < c < x$) and the quotient on the right approach [by (8.14)]. Hence $f(x)/g(x)$ also approaches L and the theorem is proved.

8.10 Exercises

Evaluate the limits in Exercises 1 through 23. The letters a and b denote positive constants.

1. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$.
2. $\lim_{x\to 2} \dfrac{3x^2 + 2x - 16}{x^2 - x - 2}$.
3. $\lim_{x\to 0} \dfrac{\log(\cos ax)}{\log(\cos bx)}$.
4. $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$.

$\lim_{x\to 0+} \frac{x - \sin x}{(x \sin x)^{3/2}}$.

$\lim_{x\to 0} \frac{\sin x}{\arctan x}$.

$\lim_{x\to 0} \frac{a^x - 1}{b^x - 1}$, $b \neq 1$.

$\lim_{x\to 1} \frac{\log x}{x^2 + x - 2}$.

$\lim_{x\to 0} \frac{1 - \cos x^2}{x^2 \sin x^2}$.

$\lim_{x\to 0} \frac{x(e^x + 1) - 2(e^x - 1)}{x^3}$.

$\lim_{x\to 0} \frac{\log (1 + x) - x}{1 - \cos x}$.

$\lim_{x\to \frac{1}{2}\pi} \frac{\cos x}{x - \frac{1}{2}\pi}$.

$\lim_{x\to 1} \frac{[\sin (\pi/2x)](\log x)}{(x^3 + 5)(x - 1)}$.

$\lim_{x\to 0} \frac{\cosh x - \cos x}{x^2}$.

15. $\lim_{x\to 0} \frac{a^x - a^{\sin x}}{x^3}$.

16. $\lim_{x\to 0} \frac{\cos (\sin x) - \cos x}{x^4}$.

17. $\lim_{x\to a+} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}}$.

18. $\lim_{x\to 0} \frac{3 \tan 4x - 12 \tan x}{3 \sin 4x - 12 \sin x}$.

19. $\lim_{x\to 1+} \frac{x^x - x}{1 - x + \log x}$.

20. $\lim_{x\to 0} \frac{\arcsin 2x - 2 \arcsin x}{x^3}$.

21. $\lim_{x\to 0} \frac{x \cot x - 1}{x^2}$.

22. $\lim_{x\to 1} \frac{\sum_{k=1}^{n} x^k - n}{x - 1}$.

23. $\lim_{x\to 0+} \frac{1}{x\sqrt{x}}\left(a \arctan \frac{\sqrt{x}}{a} - b \arctan \frac{\sqrt{x}}{b}\right)$.

For what value of the constant a will $x^{-2}(e^{ax} - e^x - x)$ tend to a finite limit as $x \to 0$? is the value of this limit?

Find constants a and b such that

$$\lim_{x\to 0} \frac{1}{bx - \sin x} \int_0^x \frac{t^2\, dt}{\sqrt{a + t}} = 1 .$$

The symbols $+\infty$ and $-\infty$. Extension of L'Hôpital's rule

Hôpital's rule may be extended in several ways. First of all, we may wish to consider quotient $f(x)/g(x)$ as x increases without bound. It is convenient to have a short ptive symbolism to express the fact that we are allowing x to increase indefinitely. his purpose mathematicians use the special symbol $+\infty$, called "plus infinity." ugh we shall not attach any meaning to the symbol $+\infty$ *by itself*, we shall give e definitions of various statements involving this symbol.

e of these statements is written as follows:

$$\lim_{x\to +\infty} f(x) = A ,$$

s read "The limit of $f(x)$, as x tends to plus infinity, is A." The idea we are trying ress here is that the function values $f(x)$ can be made arbitrarily close to the real er A by taking x large enough. To make this statement mathematically precise we explain what is meant by "arbitrarily close" and by "large enough." This is done by s of the following definition:

NITION. The symbolism

$$\lim_{x\to +\infty} f(x) = A$$