Now

$$R'(\theta) > 0 \iff \frac{\csc\theta}{r_1^4} > \frac{\cot\theta}{r_2^4} \iff \cos\theta < \frac{r_2^4}{r_1^4}$$

and

$$R'(\theta) < 0 \quad \text{when} \quad \cos\theta > \frac{r_2^4}{r_1^4}$$

Noting that $\cos\theta$ is a decreasing function of θ for $0 < \theta < \pi$, we conclude from the First Derivative Test that the resistance has an absolute minimum value when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) When $r_2 = \frac{2}{3}r_1$ we have $\cos\theta = \left(\frac{2}{3}\right)^4$ and so

$$\theta = \cos^{-1}\left(\tfrac{2}{3}\right)^4 \approx 79^\circ$$

■

EXERCISES 4.4

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.

(a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.

First number	Second number	Product
1	22	22
2	21	42
3	20	60
⋮	⋮	⋮

(b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Photosynthesis The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

☆ **7. Crop yield** A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where k is a positive constant. What nitrogen level gives the best yield?

8. The **measles pathogenesis** function

$$f(t) = -t(t - 21)(t + 1)$$

is used in Section 5.1 to model the development of the disease, where t is measured in days and $f(t)$ represents the number of infected cells per milliliter of plasma. What is the peak infection time for the measles virus?

9. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

(c) Write an expression for the total area.

(d) Use the given information to write an equation that relates the variables.

(e) Use part (d) to write the total area as a function of one variable.

(f) Finish solving the problem and compare the answer with your estimate in part (a).

10. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

(c) Write an expression for the volume.

(d) Use the given information to write an equation that relates the variables.

(e) Use part (d) to write the volume as a function of one variable.

(f) Finish solving the problem and compare the answer with your estimate in part (a).

11. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

12. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

13. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.

(b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

14. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

15. Find the point on the line $y = 2x + 3$ that is closest to the origin.

16. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

17. Age and size at maturity Most organisms grow for a period of time before maturing reproductively. For many species of insects and fish, the later the age a at maturity, the larger the individual will be, and this translates into a greater reproductive output. At the same time, however, the probability of surviving to maturity decreases as the age of maturity increases. These contrasting effects can be combined into a single measure of reproductive success in different ways. Suppose μ is a constant representing mortality rate. Find the optimal age at maturity for the following models.

(a) $r = \dfrac{\ln(ae^{-\mu a})}{a}$ (b) $R = ae^{-\mu a}$

Source: Adapted from D. Roff, *The Evolution of Life Histories: Theory and Analysis* (New York: Chapman and Hall, 1992).

18. Enzootic stability Suppose the rate at which people of age a get infected with a pathogen is given by $\lambda e^{-\lambda a}$, where λ is a positive constant. Not all infections develop into disease. Suppose that, of those individuals of age a that get infected, a fraction pa develop the disease (where p is a number chosen so that $0 < pa < 1$ over the ages of interest). At what age is the rate of disease development the highest?

Source: Adapted from P. Coleman et al., "Endemic Stability—A Veterinary Idea Applied to Public Health." *The Lancet* 357 (2001): 1284–86.

19. If $C(x)$ is the cost of producing x units of a commodity, then the **average cost** per unit is $c(x) = C(x)/x$. The **marginal cost** is the rate of change of the cost with respect to the number of items produced, that is, the derivative $C'(x)$.

(a) Show that if the average cost is a minimum, then the marginal cost equals the average cost.

(b) If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, in dollars, find (i) the cost, average cost, and marginal cost at a production level of 1000 units; (ii) the production level that will minimize the average cost; and (iii) the minimum average cost.

20. If $R(x)$ is the revenue that a company receives when it sells x units of a product, then the **marginal revenue function** is the derivative $R'(x)$. The profit function is

$$P(x) = R(x) - C(x)$$

where C is the cost function from Exercise 19.

(a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.

(b) If $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $R(x) = 1700x - 7x^2$ is the revenue function, find the production level that will maximize profit.

21. Sustainable harvesting Example 5 was based on the assumption that we want to maximize the total harvest size H, but instead we might want to maximize profit.

(a) Suppose the selling price of a unit of harvest is p dollars and the cost per unit harvested is C dollars. (Assume $p > C$.) Show that the fishing effort that maximizes profit is the same as the effort that maximizes the harvest size H.

(b) Suppose the selling price of a unit of harvest is p dollars and the unit cost is inversely proportional to the fishing effort h (that is, $C = \alpha/h$). Assume $\alpha < rp$. What is the fishing effort that maximizes profit? What is the limiting population size, assuming this fishing effort is used?