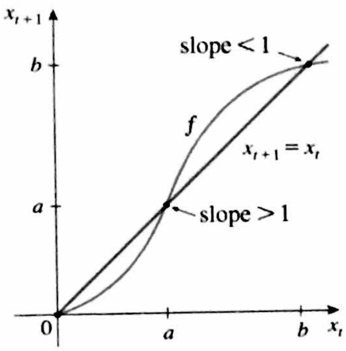

FIGURE 5

■ Stability Criterion

An equilibrium occurs when the graph of f crosses the diagonal line, which has slope 1. Figure 5 shows the increasing function f from Figure 3 and we see that at the stable equilibrium $\hat{x} = b$ the curve crosses the diagonal from above to below, so $f'(b) < 1$. At the unstable equilibrium $\hat{x} = a$ the curve crosses the diagonal from below to above, so $f'(a) > 1$.

If f is decreasing, we see from diagrams like Figure 4 that stable spirals occur when $-1 < f'(\hat{x}) < 0$ and unstable spirals occur for steeper curves, that is, $f'(\hat{x}) < -1$.

To summarize, our intuition tells us that equilibria are stable when $-1 < f'(\hat{x}) < 1$ and unstable when $f'(\hat{x}) > 1$ or $f'(\hat{x}) < -1$. So the following theorem appears plausible. A proof, using the Mean Value Theorem, appears in Appendix E.

> **(3) The Stability Criterion for Recursive Sequences** Suppose that $\hat{x}$ is an equilibrium of the recursive sequence $x_{t+1} = f(x_t)$, where f' is continuous. If $|f'(\hat{x})| < 1$, the equilibrium is stable. If $|f'(\hat{x})| > 1$, the equilibrium is unstable.

Let's revisit some of the difference equations we studied in Section 2.1 and see how the Stability Criterion applies to those equations.

EXAMPLE 3 | BB **Drug concentration** In Example 2.1.5 we considered the difference equation

$$C_{n+1} = 0.3C_n + 0.2$$

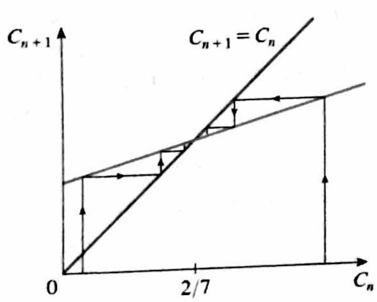

FIGURE 6

where C_n is the concentration of a drug in the bloodstream of a patient after injection on the nth day, 30% of the drug remains in the bloodstream the next day, and the daily dose raises the concentration by 0.2 mg/mL.

Here the recursion is of the form $C_{n+1} = f(C_n)$, where $f(x) = 0.3x + 0.2$. The equilibrium concentration is $\hat{C}$, where $0.3\hat{C} + 0.2 = \hat{C}$. Solving this equation gives $\hat{C} = \frac{2}{7}$. The derivative of f is $f'(\hat{C}) = 0.3$, which is less than 1, so the equilibrium is stable, as illustrated by the cobwebbing in Figure 6. In fact, in Section 2.1 we calculated that

$$\lim_{n\to\infty} C_n = \frac{2}{7}$$ ■

EXAMPLE 4 | BB **Logistic difference equation** In Example 2.1.8 we examined the long-term behavior of the terms defined by the logistic difference equation

$$x_{t+1} = cx_t(1 - x_t)$$

for different positive values of c. Use the Stability Criterion to explain that behavior.

SOLUTION We can write the logistic equation as $x_{t+1} = f(x_t)$, where

$$f(x) = cx(1 - x)$$

We first find the equilibria by solving the equation $f(x) = x$:

$$cx(1 - x) = x \iff x = 0 \quad \text{or} \quad c(1 - x) = 1$$

So one equilibrium is $\hat{x} = 0$. To find the other one, note that

$$c - cx = 1 \iff c - 1 = cx \iff x = \frac{c - 1}{c} = 1 - \frac{1}{c}$$