NOTEST ON ITERATION, NEWTON FIXED POINT

An iteration scheme is simply a function, y = F(x), but thought of as a RULE for moving points x on the real line. The iteration scheme is written

$$x_{i+1} = F(x_i) \tag{1}$$

or, as May writes it

$$X_{t+1} = F(X_t).$$

Another word for "iteration scheme" is "map" or "mapping". The index here, i or t is to be thought of as discrete time, eg. a single generation in a varying population of animals.

The orbit of a map is any series $x_1, x_2, \ldots, x_i, \ldots$ of points satisfing the requirement of eq (1): in other words the next element in the series is the map F applied to its predecessor. We also call such a sequence the ORBIT OF x_1 and refer to x_1 as the "seed" or "initial condition" or "inital guess" which generates the orbit.

A FIXED POINT x_* of mapping is a seed $x_* = x_1$ whose orbit is $x_*, x_*, x_*, \ldots$ This is the same as saying that

$$x_* = F(x_*) \tag{2}$$

The fixed point is called "stable" if every point x_1 sufficiently close to x_* satisfies that its orbit $x_1, x_2, \ldots$ tends to x_*; that is, if $\lim_i x_i \to x_*$. The fixed point is called "unstable" if the orbit of any point close to x_* but not equal to x_* eventually leaves x_*, and moves at least a fixed distance away.

THEOREM. [STABILITY TEST] The fixed point x_* for F is stable if $|F'(x_*)| < 1$. The fixed point point x_* for F is unstable if $|F'(x_*)| > 1$.

Newton iteration, derived by drawing tangent lines to graphs, is the iteration scheme used for finding roots of a function $f(x)$. A "root" of a function is the same as a "zero" which is the same as a solution to the equation $f(x) = 0$. We worked out the Newton interation to be given by the function $F(x) = x - f(x)/f'(x)$. When we want to emphasize that the Newton iteration map DEPENDS on the function f whose root we are trying to find we will write it as F_f.

EXERCISE: x_* is a fixed point of the map $F = F_f$ for Newton iteration if and only if x_* is a root of f.

EXERCISE. If x_* is a fixed point of the Newton iteration map F_f then $F'(x_*) = 0$, so the fixed points are ALWAYS stable.

By completing these exercises and then looking back at the THEOREM you will see that you have proved: the FIXED POINTS FOR the NEWTON ITERATION MAP F_f for finding roots of a function f are ALWAYS STABLE.