NOTEST ON ITERATION, NEWTON FIXED POINT

Last time we listed the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 89, \ldots$$

figured out the rule that generated it

$$F_{i+2} = F_{i+1} + F_i$$

provided that we seed it by taking $F_0 = 1, F_1 = 1$. We noted that $F_i \to +\infty$ as $i \to \infty$ but that the RATIO of successive Fibonacci numbers, F_i/F_{i+1} has a limit. To understand this limit we worked out

$$F_0/F_1 = 1/1.$$

$$F_1/F_2 = \frac{1}{1+1}$$

$$F_2/F_3 = \frac{1}{1+\frac{1}{1+1}}$$

and then we proved by induction that

$$F_{i-1}/F_i = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

with $i+2$ of the number "1's occuring in all in this last rather strange fraction. Continuing for ever, we get an infinite continued fraction. Yep. That's the name. What does it represent? It is a famous number called the Golden Mean, with whole books written about it.

To understand why and how this ratio of Fibonacci numbers converges to the Golden mean we worked out in class on Feb 19 2020 that the map

$$G(x) = \frac{1}{1+x},$$

thought of as an iteration scheme, and seeded with $x_1 = 1$ has for orbit F_{i-1}/F_i the Fibonacci ratios. Indeed, $G(x) = \frac{1}{1+1}, G(\frac{1}{1+1}) = \frac{1}{1+\frac{1}{1+1}}$, etc. so that $G^i(1) = F_i/F_{i+1}$ If this orbit converges, it MUST converge to a fixed point for G.

Definition 0.1. *I will call this $G(x)$ the Fibonacci map*

I am using $G(x)$ instead of $F(x)$ for the map so as to not confuse the Fibonacci numbers typically denoted by F_i like we have, with the map we using. So, solve the fixed point equation:

$$x = G(x)$$

which says $x = 1/(1+x)$ or $x(1+x) = 1$ or finally

$$x^2 + x - 1 = 0$$

This is a quadratic equation. Its roots are

$$x_{\pm} = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{1}{2}(-1 \pm \sqrt{5}).$$

One of these roots x_+ is positive, the other is negative. Their product is -1 and their sum is $+1$. Either the positive one x_+, or its reciprocal (which is $|x_-|$), is called the GOLDEN MEAN.

We use our STABILITY CONDITION THEOREM to verify that the positive root x_+ is a stable fixed point, which basically guarantees that the orbit above, i..e the Fibonacci ratios, converge to the Golden mean. Well,

$$G'(x) = \frac{-1}{(1+x)^2}$$

We are to figure out whether or not $|G'(x_+)| < 1$. We have that $|G'(x)| = \frac{1}{(1+x)^2}$ which is less than 1 for any positve number x. Since $x_+ > 1$ we have that $|G'(x_+)| < 1$. YES. This fixed point is stable. We are attracted to the golden mean.

GRAPH HERE. Check that other root x_- is an UNSTABLE fixed point for the Fibonacci map.

EXERCISE. Work out the exact value of $|F'(x_+)|$. Your answer will involve $\sqrt{5}$.

TANGENTIAL REMARKS. CONTRACTION MAPPINGS. Some algebra for you.

Exercise. Compute $G(x) - G(y)$ for two numbers x, y.

Exercise. Verify that $|G(x) - G(y)| \leq \frac{|x-y|}{(1+x)(1+y)}$, for $x, y > 0$.

DEFINITION. A mapping $G(x)$ defined on an interval of real numbers is called a CONRACTION MAPPING if there is a number C, with $C < 1$ such that $|G(x) - G(y)| \leq C|x - y|$ whenever x, y is in the interval.

PICTURE HERE of Meaning of contraction mapping.

EXERCISE. Show that our Fibonacci mapping is a contraction mapping when restricted to the interval $[1/11, 10]$, and that we can take C to equal $(\frac{1}{1+\frac{1}{11}})^2$.

THEOREM. A contraction mapping applied to a closed bounded interval has a unique fixed point on that interval.

Sketch proof. Pick any point in the interval as seed. Its orbit under G limits to the fixed point.