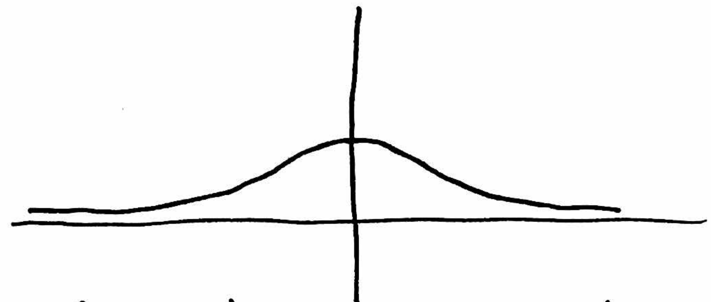
Chain rule.

Example

e^{-x^2} :



(Bell curve of probability & statistics, aka "Gaussian".

$$e^{-x^2} = f(g(x))$$
$$= f(-x^2)$$
$$= e^{-x^2}$$

So: $g(x) = -x^2 (= u)$.
$f(u) = e^u$

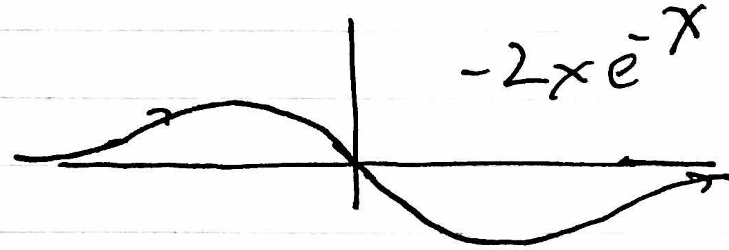
CHAIN RULE

$$\boxed{f(g(x))' = f'(g(x)) \cdot g'(x)}$$

Us: $f'(u) = e^u$; $g'(x) = -2x$.

$$f'(g(x))\, g'(x) = (e^{-x^2})(-2x)$$
$$= -2x\, e^{-x^2}.$$

Picture $-2xe^{-x^2}$.



odd fn . . .

Another example

$$\sqrt{x^2+1} = f(g(x))$$
$$= \sqrt{g(x)}$$
$$= \sqrt{x^2+1}$$

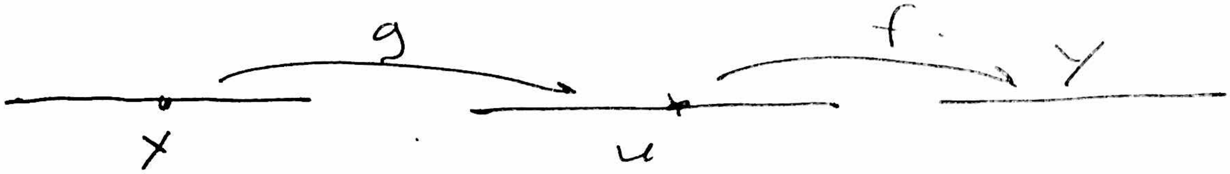
so $f(u) = \sqrt{u}$; $f'(u) = \frac{1}{2}u^{-1/2}$.

$g(x) = x^2+1$, $g'(x) = 2x$.

$$f'(g(x))\,g'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$$
$$= \frac{x}{\sqrt{x^2+1}}.$$

Another way to do chain rule & remember it.

$$x \longmapsto g(x) = u \longmapsto f(u) = f(g(x))$$



$$\boxed{\text{if } y = f(g(x)) \ \& \ u = g(x) \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.}$$

CHAIN RULE!

e^{-x^2} again; $\underbrace{u = -x^2}$; $y = e^u$.

$$\frac{dy}{du}\frac{du}{dx} = e^u(-2x)$$

Now plug in

get $e^{-x^2}(-2x)$

or $-2x\, e^{-x^2}$.

Why does it work?

1st for linear functions.

eg. $f(g(x))$.

$g(x) = 5x + 1$

$f(x) = 3x - 11$

$g'(x) = ?$

$f'(x) = ?$

Why???

$f(g(x))$ is a linear fn w/ slope 15. since $3 \cdot 5 = 15$

so $f(g(x))' = 15$.

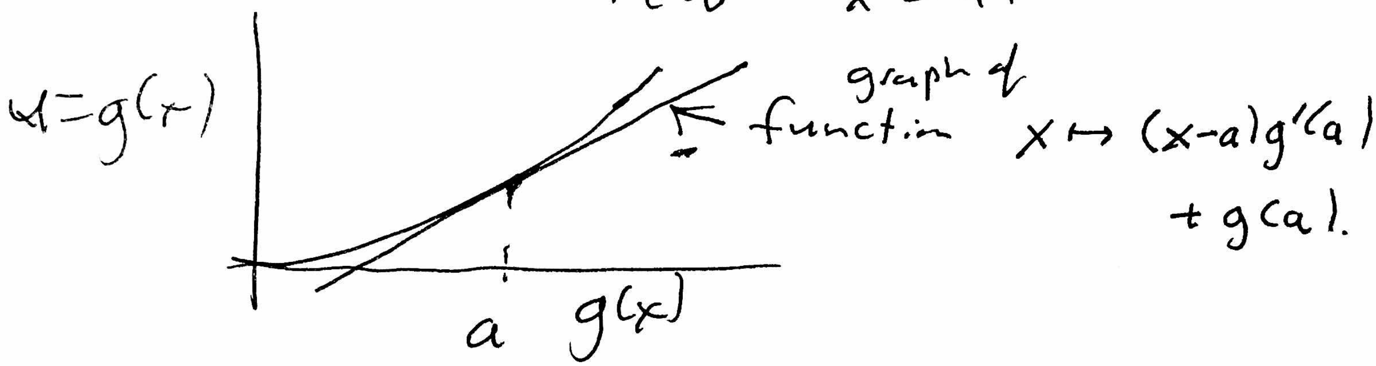
check: $f(g(x)) = 5(3x - 11) + 1$

$= 15x - 55 + 1$

$f(g(x))' = 15$.

Def of derivative of $g(x)$: ~~Best line~~ at $x = a$.

<u>Best</u> linear approx to $g(x)$ Near $x = a$.



$f'(g(a))\,g'(a))$ = ~~the~~ product of slopes of corresp. <u>best</u> linear approx.

A few others:

$$\frac{d}{dx}(x+1)^{100}$$

$$u = x+1; \quad f(u) = u^{100} = y$$

$$\frac{d}{d(x+1)}\left((x+1)^{100}\right)\frac{d(x+1)}{dx}$$

is

$$\frac{dy}{du}\frac{du}{dx} \quad \therefore \quad 100(x+1)^{99}\cdot 1.$$

You *could* do it by expanding out & using the power rule

$$(x+1)^{100} = x^{100} + 100x^{99} + \binom{100}{2}x^{98} + \binom{100}{3}x^{97} + \cdots + 100x + 1.$$

But that is so much more painful!

& a reality check with the power rule:

$$x^{15} = (x^3)^5$$

$$= f(g(x))$$

with $g(x) = x^3$

$f(u) = u^5.$

On the one hand

$$\boxed{(x^{15})' = 15x^{14}.}$$

on the other hand,

$$f'(g(x))\,g'(x) = 5(x^3)^4 \cdot 3x^2$$

$$= 5x^{12} \cdot 3x^2$$

$$= 15x^{14} \checkmark$$

Yes, it checks out!

8

Finally, from e^{Kx}

K a constant

$$\frac{d}{dx}\left(e^{Kx}\right) = \frac{d}{du}\left(e^{u}\right)\frac{du}{dx}$$

with $\boxed{u = Kx}$

$$= e^{u} \cdot K$$

$$= K e^{Kx}$$

More generally:

$$\frac{d}{dx} f(Kx) = K \frac{df}{dx}(Kx)$$

Other generalities:

$$\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$$

$$\frac{d}{dx}\left(f(x)^{N}\right) = N f(x)^{N-1} f'(x)$$

etc etc etc.

9 Eg quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

First: $\frac{1}{g(x)} = g(x)^{-1}$

So by That last rule:

$$\left(\frac{1}{g}\right)' = -g(x)^{-2}g'(x) = \frac{-g'}{g^2}.$$

Then $\left(\frac{f}{g}\right) = f \cdot \left(\frac{1}{g}\right)$ so by the product rule:

$$\left(\frac{f}{g}\right)' = f'\left(\frac{1}{g}\right) + f\left(\frac{1}{g}\right)'$$

$$= \frac{f'}{g} + f\left(\frac{-g'}{g^2}\right)$$

$$= \frac{f'}{g} - \frac{fg'}{g^2}$$

$$= \frac{f'g}{g^2} - \frac{fg'}{g^2}$$

$$= \frac{f'g - fg'}{g^2}.$$