■ PROJECT The Tragedy of the Commons: An Introduction to Game Theory

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In Example 4.4.5 we explored sustainable fish harvesting. We assumed that a single company is exploiting the resource and found that the steady-state population size in the presence of harvesting satisfied the equation

$$rN\left(1 - \frac{N}{K}\right) = hN$$

where N is the population size, r and K are positive constants, and h is the fishing effort. In reality, fish stocks are part of the "Commons," meaning that no single person has exclusive rights to them. Suppose, for example, that a second company begins to exploit the same population. Then there are two fishing efforts, h_1 and h_2, one for each company. Once the population size has stabilized, the equation

$$rN\left(1 - \frac{N}{K}\right) = h_1N + h_2N$$

must hold, where h_1N and h_2N are the total harvests for companies 1 and 2, respectively. Suppose you run company 1 and before company 2 arrives you are using the optimal h calculated in Example 4.4.5, that is, $h_1 = \frac{1}{2}r$.

1. When company 2 arrives, it needs to decide upon a fishing effort h_2. What value of h_2 maximizes its harvest once the population has reached a steady state, assuming that you continue using h_1?

2. Once your competitor is using their rate obtained in Problem 1, your harvesting rate will no longer be optimal for you. What is your new optimal rate h_1^*, given that your competitor continues to use the rate found in Problem 1?

3. More generally, determine your optimal fishing effort as a function of the rate your competitor uses and your competitor's optimal fishing effort as a function of the rate you use. These are referred to as the "best response" fishing efforts.

4. The harvesting problem can be viewed as a game played between the two companies, where the payoff to each depends on both of their choices of fishing effort. An area of mathematics called **game theory** has been developed to analyze such problems. A key concept in game theory is that of a **Nash equilibrium**, which is a pair of values h_1^* and h_2^* that simultaneously satisfy both best response functions. At a Nash equilibrium each party is doing the best that it can, given the choice of its competitor. What is the Nash equilibrium pair of fishing efforts?

5. What is the total population size at the Nash equilibrium, and what are the total harvests of you and your competitor?

6. Demonstrate that both you and your competitor could have a higher total harvest than that attained at the Nash equilibrium if you could agree to cooperate and to split the harvest that you were obtaining before your competitor showed up.

7. Problem 6 shows that both you and your competitor are worse off at the Nash equilibrium than you would be if you agreed to cooperate. Show that, in terms of population size, the fish population is also worse off. This is the "Tragedy of the Commons."

Nash

John F. Nash, Jr. (1928–) is an American mathematician best known for his work in game theory. He developed the idea now known as a Nash equilibrium in his 28-page doctoral thesis in 1950. In 1994 he was awarded the Nobel Prize in Economics for this work. Nash also made several other foundational contributions to advanced mathematics, despite suffering from schizophrenia. His extraordinary life is chronicled in the book *A Beautiful Mind* by Sylvia Nasar and in a Hollywood movie of the same name.

4.5 Recursions: Equilibria and Stability

In Section 1.6 we looked at recursive sequences, which we also called difference equations or discrete-time models. These are defined by a recursion of the form

$$a_{n+1} = f(a_n) \qquad \text{or} \qquad x_{t+1} = f(x_t) \qquad \text{or} \qquad N_{t+1} = f(N_t)$$

where f is the updating function, N_t is the number of individuals in a population at time t, and N_{t+1} is the population one unit of time into the future. Then in Section 2.1 we investigated the long-term behavior of such recursions. In particular, we saw that some recursive sequences approach a limiting value as t becomes large:

$$\lim_{t \to \infty} x_t = L$$

Here we assume that the updating function f that defines the recursion is a differentiable function and learn that the values of its derivative play a role in determining the limiting behavior of the sequence.

■ Equilibria

(1) Definition An **equilibrium** of a recursive sequence $x_{t+1} = f(x_t)$ is a number $\hat{x}$ that is left unchanged by the function f, that is,

$$f(\hat{x}) = \hat{x}$$

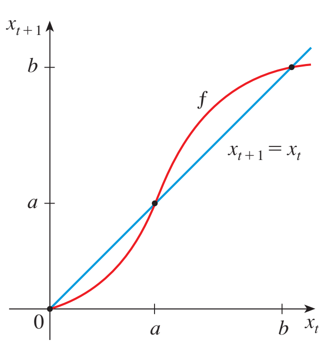


FIGURE 1
The recursion $x_{t+1} = f(x_t)$ has three equilibria

It's helpful to think of an equilibrium as a point on a number line. An equilibrium is sometimes called a **fixed point** because f leaves the point $\hat{x}$ fixed. Notice that if $\hat{x}$ is an equilibrium and if, for instance, $x_6 = \hat{x}$, then

$$x_7 = f(x_6) = f(\hat{x}) = \hat{x}$$

and, similarly, all of the following terms in the sequence are also equal to $\hat{x}$.

To find the equilibria algebraically, we solve the equation $f(x) = x$, if possible. To locate them geometrically we graph the curves $y = f(x)$ and $y = x$ (the diagonal line) and see where they intersect. Because the recursion is $x_{t+1} = f(x_t)$, when we graph f we label the horizontal and vertical axes x_t and x_{t+1}, as in Figure 1. For that particular recursion we see that there are three points of intersection and therefore three equilibria, 0, a, and b.

(2) Definition An equilibrium is called **stable** if solutions that begin close to the equilibrium approach that equilibrium. It is called **unstable** if solutions that start close to the equilibrium move away from it.

So when we say that $\hat{x}$ is a stable equilibrium of the recursion $x_{t+1} = f(x_t)$ we mean that if x_t is a solution of the recursion and x_0 is sufficiently close to $\hat{x}$, then $x_t \to \hat{x}$ as $t \to \infty$.

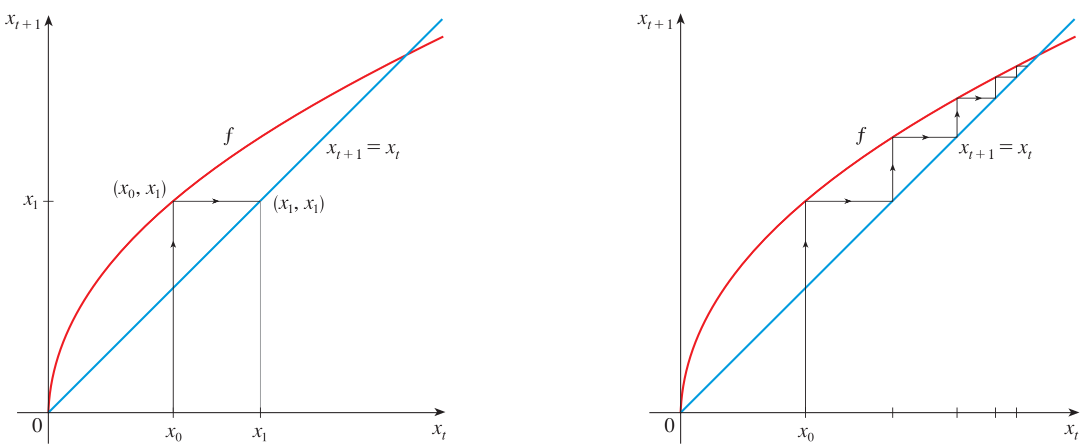
EXAMPLE 1 | Determine the equilibrium of the difference equation $N_{t+1} = RN_t$, where $R > 0$, and classify it as stable or unstable.

SOLUTION The equilibrium $\hat{N}$ satisfies the equation $\hat{N} = R\hat{N}$. The only solution of $(R - 1)\hat{N} = 0$ is $\hat{N} = 0$, unless $R = 1$. We know that the solution of the recursion $N_{t+1} = RN_t$ is $N_t = N_0 \cdot R^t$. There are three cases:

- If $0 < R < 1$, then $N_t = N_0 \cdot R^t \to 0$ as $t \to \infty$, so $N_t \to \hat{N} = 0$. Therefore the equilibrium $\hat{N} = 0$ is stable in this case.
- If $R > 1$, then $N_t = N_0 \cdot R^t \to \infty$ as $t \to \infty$, and so the equilibrium $\hat{N} = 0$ is unstable in this case.
- If $R = 1$, then $N_t = N_0$ for all t. This case is called *neutral*. ■

■ Cobwebbing

There is a graphical method for finding equilibria and determining whether they are stable or unstable. It is called **cobwebbing** and is illustrated in Figure 2. We start with an initial value x_0 on the horizontal axis and locate $x_1 = f(x_0)$ as the distance from the point x_0 up to the point (x_0, x_1) on the graph of f. Then we draw the horizontal line segment from (x_0, x_1) to the point (x_1, x_1) on the diagonal line. The point x_1 lies directly beneath (x_1, x_1) on the horizontal axis.



(a) (b)

FIGURE 2
Cobwebbing

In Figure 2(b) we repeat this procedure to obtain x_2 from x_1, drawing a vertical line segment from (x_1, x_1) to (x_1, x_2) on the graph of f and then a horizontal line segment over to the diagonal. Continuing in this manner we create a zigzag path that reflects off the diagonal line and shows how the successive terms in the sequence can be obtained geometrically.

EXAMPLE 2 | Use cobwebbing to determine whether the equilibria $\hat{x} = 0$, $\hat{x} = a$, and $\hat{x} = b$ in Figure 1 are stable or unstable.

SOLUTION Figure 3 is a larger version of Figure 1. We experiment with different initial values and use cobwebbing to visualize the values of x_t. We notice that

$$\text{if } a < x_0 < b, \text{ then } \lim_{t\to\infty} x_t = b$$

$$\text{but} \quad \text{if } 0 < x_0 < a, \text{ then } \lim_{t\to\infty} x_t = 0$$

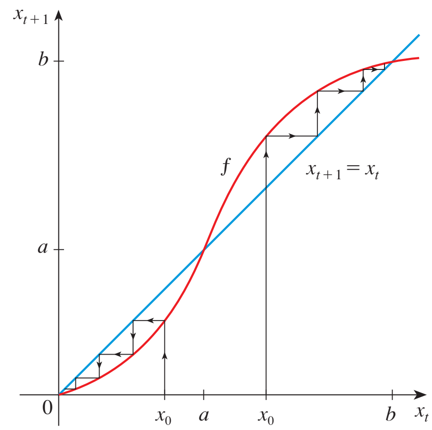


FIGURE 3
Cobwebbing with stable and unstable equilibria

Solutions that start close to b approach b, so $\hat{x} = b$ is a stable equilibrium. Likewise, solutions that start close to 0 approach 0, so $\hat{x} = 0$ is also a stable equilibrium. But solutions that start close to a (on either side of a) move away from a. So $\hat{x} = a$ is an unstable equilibrium. ■

So far we have used cobwebbing only with increasing functions f. Figure 4 shows what happens when f decreases. We apply cobwebbing with initial value x_0 to a difference equation $x_{t+1} = f(x_t)$ with decreasing f. Instead of the zigzag paths in Figures 2 and 3, you can see that we get spiral paths and the values of x_t oscillate around the equilibrium $\hat{x}$. In Figure 4(a), $x_t \to \hat{x}$ as $t \to \infty$, so $\hat{x}$ is stable. In Figure 4(b), however, the values of x_t move away from $\hat{x}$, so $\hat{x}$ is unstable.

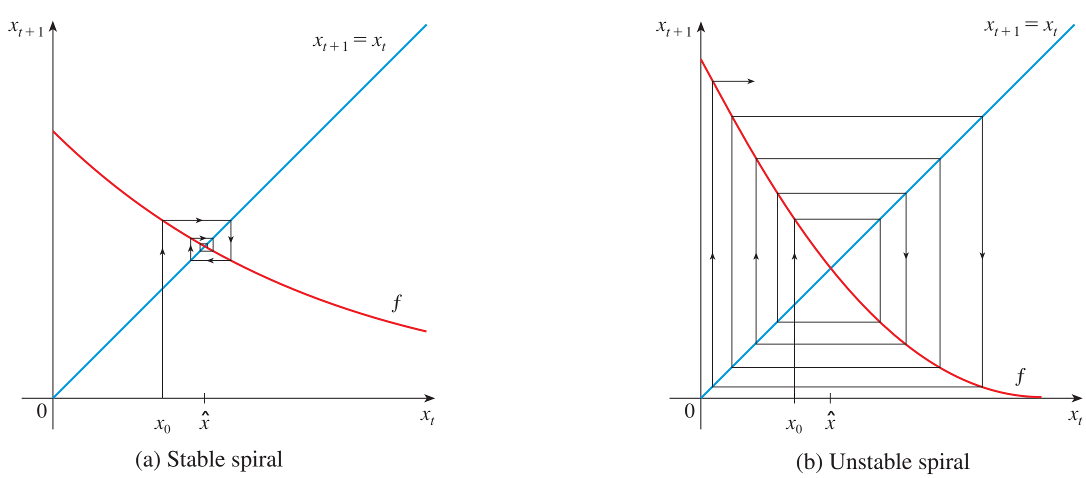


FIGURE 4 Spiral cobwebbing

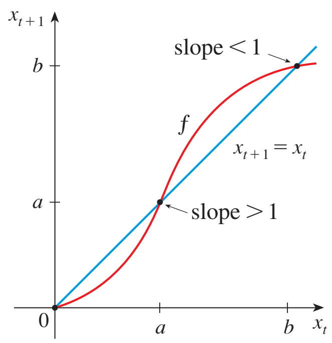


FIGURE 5

■ Stability Criterion

An equilibrium occurs when the graph of f crosses the diagonal line, which has slope 1. Figure 5 shows the increasing function f from Figure 3 and we see that at the stable equilibrium $\hat{x} = b$ the curve crosses the diagonal from above to below, so $f'(b) < 1$. At the unstable equilibrium $\hat{x} = a$ the curve crosses the diagonal from below to above, so $f'(a) > 1$.

If f is decreasing, we see from diagrams like Figure 4 that stable spirals occur when $-1 < f'(\hat{x}) < 0$ and unstable spirals occur for steeper curves, that is, $f'(\hat{x}) < -1$.

To summarize, our intuition tells us that equilibria are stable when $-1 < f'(\hat{x}) < 1$ and unstable when $f'(\hat{x}) > 1$ or $f'(\hat{x}) < -1$. So the following theorem appears plausible. A proof, using the Mean Value Theorem, appears in Appendix E.

> **(3) The Stability Criterion for Recursive Sequences** Suppose that $\hat{x}$ is an equilibrium of the recursive sequence $x_{t+1} = f(x_t)$, where f' is continuous. If $|f'(\hat{x})| < 1$, the equilibrium is stable. If $|f'(\hat{x})| > 1$, the equilibrium is unstable.

Let's revisit some of the difference equations we studied in Section 2.1 and see how the Stability Criterion applies to those equations.

EXAMPLE 3 | **BB** Drug concentration In Example 2.1.5 we considered the difference equation

$$C_{n+1} = 0.3C_n + 0.2$$

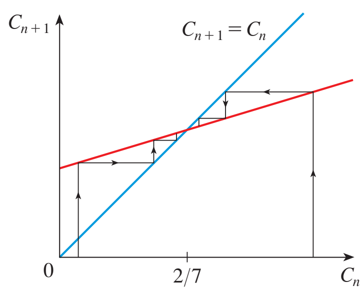


FIGURE 6

where C_n is the concentration of a drug in the bloodstream of a patient after injection on the nth day, 30% of the drug remains in the bloodstream the next day, and the daily dose raises the concentration by 0.2 mg/mL.

Here the recursion is of the form $C_{n+1} = f(C_n)$, where $f(x) = 0.3x + 0.2$. The equilibrium concentration is $\hat{C}$, where $0.3\hat{C} + 0.2 = \hat{C}$. Solving this equation gives $\hat{C} = \frac{2}{7}$. The derivative of f is $f'(\hat{C}) = 0.3$, which is less than 1, so the equilibrium is stable, as illustrated by the cobwebbing in Figure 6. In fact, in Section 2.1 we calculated that

$$\lim_{n\to\infty} C_n = \frac{2}{7}$$ ■

EXAMPLE 4 | **BB** Logistic difference equation In Example 2.1.8 we examined the long-term behavior of the terms defined by the logistic difference equation

$$x_{t+1} = cx_t(1 - x_t)$$

for different positive values of c. Use the Stability Criterion to explain that behavior.

SOLUTION We can write the logistic equation as $x_{t+1} = f(x_t)$, where

$$f(x) = cx(1 - x)$$

We first find the equilibria by solving the equation $f(x) = x$:

$$cx(1 - x) = x \iff x = 0 \quad \text{or} \quad c(1 - x) = 1$$

So one equilibrium is $\hat{x} = 0$. To find the other one, note that

$$c - cx = 1 \iff c - 1 = cx \iff x = \frac{c - 1}{c} = 1 - \frac{1}{c}$$

So the other equilibrium is

$$\hat{x} = 1 - \frac{1}{c}$$

The derivative of $f(x) = c(x - x^2)$ is $f'(x) = c(1 - 2x)$. For the first equilibrium, $\hat{x} = 0$, we have $f'(0) = c$, so the Stability Criterion tells us that $\hat{x} = 0$ is stable if $0 < c < 1$ and unstable if $c > 1$.

For the second equilibrium, $\hat{x} = 1 - 1/c$, we have

$$f'\left(1 - \frac{1}{c}\right) = c\left[1 - 2\left(1 - \frac{1}{c}\right)\right] = c\left(\frac{2}{c} - 1\right) = 2 - c$$

The Stability Criterion says that this equilibrium is stable if $|2 - c| < 1$. But

$$|2 - c| < 1 \iff -1 < 2 - c < 1 \iff 1 < c < 3$$

We also note that $f'(\hat{x})$ is negative when $2 - c < 0$, that is, $c > 2$, so oscillation occurs when $c > 2$. Let's compile all this information in the following chart:

	$\hat{x} = 0$	$\hat{x} = 1 - \frac{1}{c}$
$0 < c < 1$	stable	
$1 < c < 2$	unstable	stable
$2 < c < 3$	unstable	stable (oscillation)
$c > 3$	unstable	unstable (oscillation)

In Exercises 17–20 you are asked to illustrate the four cases in the chart in Example 4 both by cobwebbing and by graphing the recursive sequence.

Referring to the chart, we find an explanation for what we noticed in Example 2.1.8. When $c < 3$, one of the equilibria is stable and so the terms converge to that number. But when $c > 3$ both equilibria are unstable and so the terms have nowhere to go; they don't approach any fixed number. ■

EXAMPLE 5 | **BB** Ricker equation W. E. Ricker introduced the discrete-time model

$$x_{t+1} = cx_t e^{-x_t} \qquad c > 0$$

in the context of modeling fishery populations. Find the equilibria and determine the values of c for which they are stable.

SOLUTION The Ricker equation is $x_{t+1} = f(x_t)$, where

$$f(x) = cxe^{-x}$$

To find the equilibria we solve the equation $f(x) = x$:

$$cxe^{-x} = x \iff x = 0 \text{ or } ce^{-x} = 1$$

One equilibrium is $\hat{x} = 0$. The other satisfies

$$ce^{-x} = 1 \iff c = e^x \iff x = \ln c$$

The second equilibrium is $\hat{x} = \ln c$.

We use the Product Rule to differentiate f:

$$f'(x) = cx(-e^{-x}) + ce^{-x} = c(1 - x)e^{-x}$$

For $\hat{x} = 0$ we have $f'(0) = c$, so it is stable if $0 < c < 1$ and unstable if $c > 1$. For $\hat{x} = \ln c$ we get

$$f'(\ln c) = c(1 - \ln c)e^{-\ln c} = c(1 - \ln c) \cdot \frac{1}{c} = 1 - \ln c$$

Therefore

$$|f'(\hat{x})| < 1 \quad \Longleftrightarrow \quad |1 - \ln c| < 1 \quad \Longleftrightarrow \quad -1 < 1 - \ln c < 1$$

Now

$$1 - \ln c < 1 \quad \Longleftrightarrow \quad \ln c > 0 \quad \Longleftrightarrow \quad c > 1$$

and

$$-1 < 1 - \ln c \quad \Longleftrightarrow \quad \ln c < 2 \quad \Longleftrightarrow \quad c < e^2$$

By the Stability Criterion, $\hat{x} = \ln c$ is stable when

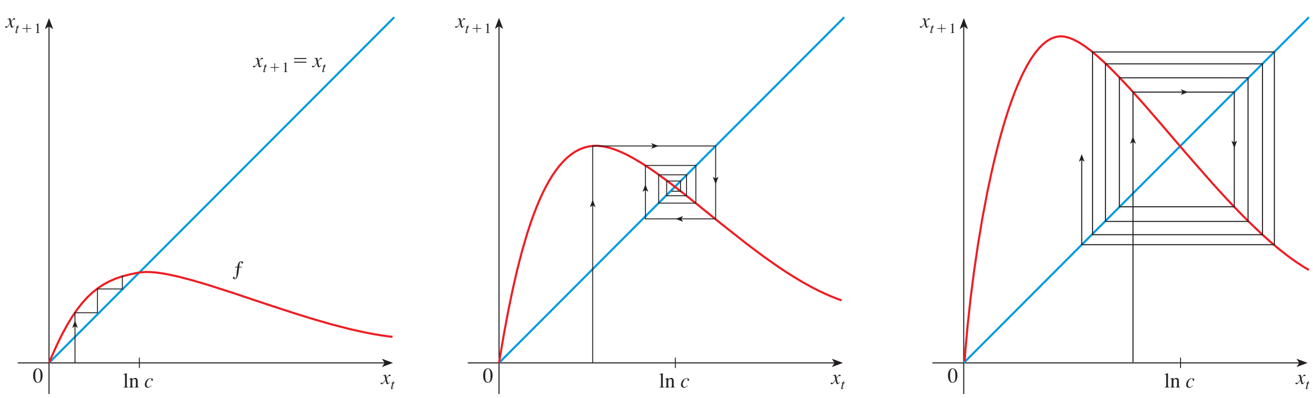
$$1 < c < e^2$$

When $0 < c < 1$ or $c > e^2$, $\hat{x} = \ln c$ is unstable.

We also note that oscillation occurs when $f'(\hat{x}) < 0$, so

$$1 - \ln c < 0 \quad \Rightarrow \quad \ln c > 1 \quad \Rightarrow \quad c > e$$

Figure 7 illustrates cobwebbing for the Ricker equation for three values of c.



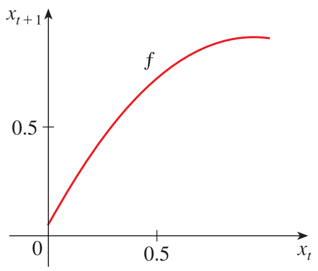
$c = 2.5$ stable | $c = 6$ stable spiral | $c = 9$ unstable spiral

FIGURE 7

EXERCISES 4.5

1–4 The graph of the function f for a recursive sequence $x_{t+1} = f(x_t)$ is shown. Estimate the equilibria and classify them as stable or unstable. Confirm your answer by cobwebbing.

1.



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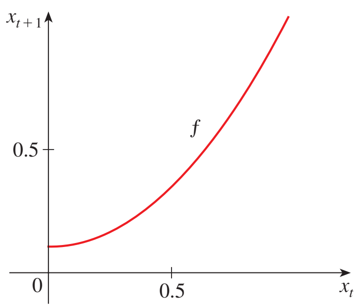

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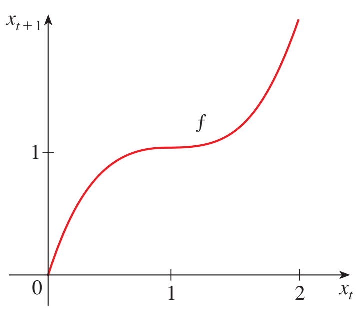

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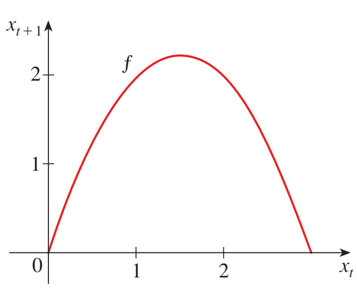


5–10 Find the equilibria of the difference equation and classify them as stable or unstable.

5. $x_{t+1} = \frac{1}{2}x_t^2$

6. $x_{t+1} = 1 - x_t^2$

7. $x_{t+1} = \dfrac{x_t}{0.2 + x_t}$

8. $x_{t+1} = \dfrac{3x_t}{1 + x_t}$

9. $x_{t+1} = 10x_t e^{-2x_t}$

10. $x_{t+1} = x_t^3 - 3x_t^2 + 3x_t$

11–12 Find the equilibria of the difference equation and classify them as stable or unstable. Use cobwebbing to find $\lim_{t\to\infty} x_t$ for the given initial values.

11. $x_{t+1} = \dfrac{4x_t^2}{x_t^2 + 3}$, $x_0 = 0.5$, $x_0 = 2$

12. $x_{t+1} = \dfrac{7x_t^2}{x_t^2 + 10}$, $x_0 = 1$, $x_0 = 3$

13–14 Find the equilibria of the difference equation. Determine the values of c for which each equilibrium is stable.

13. $x_{t+1} = \dfrac{cx_t}{1 + x_t}$

14. $x_{t+1} = \dfrac{x_t}{c + x_t}$

15. Drug pharmacokinetics A patient takes 200 mg of a drug at the same time every day. Just before each tablet is taken, 10% of the drug remains in the body.

(a) If Q_n is the quantity of the drug in the body just after the nth tablet is taken, write a difference equation expressing Q_{n+1} in terms of Q_n.
(b) Find the equilibria of the equation in part (a).
(c) Draw a cobwebbing diagram for the equation.

16. Drug pharmacokinetics A patient is injected with a drug every 8 hours. Immediately before each injection the concentration of the drug has been reduced by 40% and the new dose increases the concentration by 1.2 mg/mL.

(a) If Q_n is the concentration of the drug in the body just after the nth injection is given, write a difference equation expressing Q_{n+1} in terms of Q_n.
(b) Find the equilibria of the equation in part (a).
(c) Draw a cobwebbing diagram for the equation.

17–20 Logistic difference equation Illustrate the results of Example 4 for the logistic difference equation by cobwebbing and by graphing the first ten terms of the sequence for the given values of c and x_0.

17. $c = 0.8$, $x_0 = 0.6$

18. $c = 1.8$, $x_0 = 0.1$

19. $c = 2.7$, $x_0 = 0.1$

20. $c = 3.6$, $x_0 = 0.4$

21. Sustainable harvesting In Example 4.4.5 we looked at a model of sustainable harvesting, which can be formulated as a discrete-time model:

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) - hN_t$$

Find the equilibria and determine when each is stable.

22. Heart excitation A simple model for the time x_t it takes for an electrical impulse in the heart to travel through the atrioventricular node of the heart is

$$x_{t+1} = \frac{375}{x_t - 90} + 100 \qquad x_t > 90$$

(a) Find the relevant equilibrium and determine when it is stable.
(b) Draw a cobwebbing diagram.

Source: Adapted from D. Kaplan et al., *Understanding Nonlinear Dynamics* (New York: Springer-Verlag, 1995).

23. Species discovery curves A common assumption is that the rate of discovery of new species is proportional to the fraction of currently undiscovered species. If d_t is the fraction of species discovered by time t, a recursion equation describing this process is

$$d_{t+1} = d_t + a(1 - d_t)$$

where a is a constant representing the discovery rate and satisfies $0 < a < 1$. Find the equilibria and determine the stability.

24. Drug resistance in malaria In the project on page 78 we developed the following recursion equation for the spread of a gene for drug resistance in malaria:

$$p_{t+1} = \frac{p_t^2 W_{RR} + p_t(1 - p_t)W_{RS}}{p_t^2 W_{RR} + 2p_t(1 - p_t)W_{RS} + (1 - p_t)^2 W_{SS}}$$

where W_{RR}, W_{RS}, and W_{SS} are constants representing the probability of survival of the three genotypes. In fact this model applies to the evolutionary dynamics of any gene in a population of diploid individuals.

(a) Find the equilibria of the model in terms of the constants.
(b) Suppose that $W_{RR} = \frac{3}{4}$, $W_{RS} = \frac{1}{2}$, and $W_{SS} = \frac{1}{4}$. Determine the stability of each equilibrium (provided it lies between 0 and 1). Plot the cobwebbing diagram and interpret your results.
(c) Suppose that $W_{RR} = \frac{1}{2}$, $W_{RS} = \frac{3}{4}$, and $W_{SS} = \frac{1}{4}$. Determine the stability of each equilibrium. Plot the cobwebbing diagram and interpret your results.

25. Blood cell production A simple model of blood cell production is given by

$$R_{t+1} = R_t(1 - d) + F(R_t)$$

where d is the fraction of red blood cells that die from one day to the next and $F(x)$ is a function specifying the number of new cells produced in a day, given that the current number is x. Find the equilibria and determine the stability in each case.

(a) $F(x) = \theta(K - x)$, where θ and K are positive constants
(b) $F(x) = \dfrac{ax}{b + x^2}$, where a and b are positive constants and $a > bd$

Source: Adapted from N. Mideo et al., "Understanding and Predicting Strain-Specific Patterns of Pathogenesis in the Rodent Malaria *Plasmodium chabaudi*," *The American Naturalist* 172 (2008): E214–E328.

4.6 Antiderivatives

Suppose you know the rate at which a bacteria population is increasing and want to know the size of the population at some future time. Or suppose you know the rate of decrease of your blood alcohol concentration and want to know your BAC an hour from now. In each case, the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an *antiderivative* of f.

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$. But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore both F and G are antiderivatives of f. Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f. The following theorem says that f has no other antiderivative. A proof of Theorem 1, using the Mean Value Theorem, is outlined in Exercise 46.

(1) Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

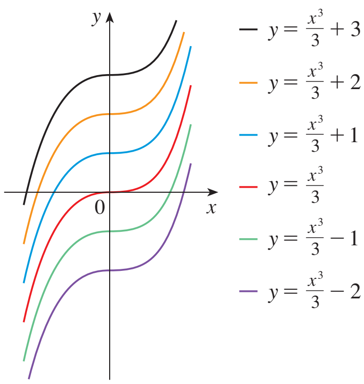


FIGURE 1
Members of the family of antiderivatives of $f(x) = x^2$

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $\frac{1}{3}x^3 + C$. By assigning specific values to the constant C, we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1). This makes sense because each curve must have the same slope at any given value of x.

EXAMPLE 1 | Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ (b) $f(x) = 1/x$ (c) $f(x) = x^n$, $n \neq -1$

SOLUTION

(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$. By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.

(b) Recall from Section 3.7 that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

So on the interval $(0, \infty)$ the general antiderivative of $1/x$ is $\ln x + C$. We also learned that

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$$

for all $x \neq 0$. Theorem 1 then tells us that the general antiderivative of $f(x) = 1/x$ is $\ln |x| + C$ on any interval that doesn't contain 0. In particular, this is true on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. So the general antiderivative of f is

$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$

(c) We use the Power Rule to discover an antiderivative of x^n. In fact, if $n \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Thus the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

This is valid for $n \geqslant 0$ since then $f(x) = x^n$ is defined on an interval. If n is negative (but $n \neq -1$), it is valid on any interval that doesn't contain 0. ■

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives. Each formula in the table is true because the derivative of the function in the right column appears in the left column. In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)

(2) Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$1/x$	$\ln \lvert x \rvert$	$\sec x \tan x$	$\sec x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
e^{cx}	$\dfrac{1}{c}e^{cx}$		

To obtain the most general antiderivative from the particular ones in Table 2, we have to add a constant (or constants), as in Example 1.

EXAMPLE 2 | Find all functions g such that

$$g'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$$

SOLUTION We first rewrite the given function as follows:

$$g'(x) = 4 \sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4 \sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

Thus we want to find an antiderivative of

$$g'(x) = 4 \sin x + 2x^4 - x^{-1/2}$$

Using the formulas in Table 2 together with Theorem 1, we obtain

$$g(x) = 4(-\cos x) + 2\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= -4 \cos x + \tfrac{2}{5}x^5 - 2\sqrt{x} + C$$ ■

In applications of calculus it is very common to have a situation as in Example 2, where it is required to find a function, given knowledge about its derivatives. An equation that involves the derivatives of a function is called a **differential equation**. These will be studied in some detail in Chapter 7, but for the present we can solve some elementary differential equations. The general solution of a differential equation involves an arbitrary constant (or constants) as in Example 2. However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

A differential equation of the form

$$\frac{dy}{dt} = f(t)$$

is called a **pure-time differential equation** because the right side of the equation does not depend on y; it depends only on t (time). The solution will be a family of antiderivatives of f. The initial value of the solution may be specified by an **initial condition** of the form $y = y_0$ when $t = t_0$. Then the problem of finding a solution of the differential equation that also satisfies the initial condition is called an **initial-value problem**:

$$\frac{dy}{dt} = f(t) \qquad y = y_0 \text{ when } t = t_0$$

Figure 2 shows the graphs of the function f' in Example 3 and its antiderivative f. Notice that $f'(x) > 0$, so f is always increasing. Also notice that when f' has a maximum or minimum, f appears to have an inflection point. So the graph serves as a check on our calculation.

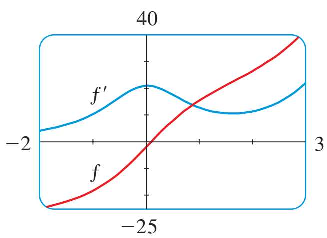


FIGURE 2

EXAMPLE 3 | Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

SOLUTION The general antiderivative of

$$f'(x) = e^x + \frac{20}{1 + x^2}$$

is

$$f(x) = e^x + 20 \tan^{-1}x + C$$

To determine C we use the fact that $f(0) = -2$:

$$f(0) = e^0 + 20 \tan^{-1} 0 + C = -2$$

Thus we have $C = -2 - 1 = -3$, so the particular solution is

$$f(x) = e^x + 20 \tan^{-1}x - 3$$ ■

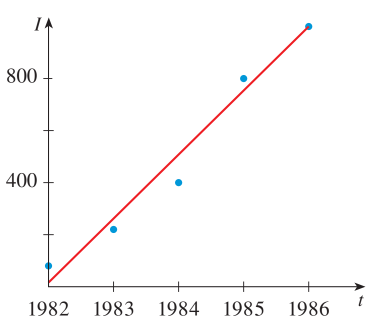


FIGURE 3

EXAMPLE 4 | HIV incidence and prevalence The rate I at which people were becoming infected with HIV (termed the incidence) in New York in the early 1980s is plotted in Figure 3. We can see from the figure that the data are well approximated by the linear function $I(t) = 16 + 242t$, where t measures the number of years since 1982. Suppose there were 80 infections at year $t = 0$. What is the number of infections expected in 1990 (termed the prevalence)?

SOLUTION Let $P(t)$ be the prevalence in year t, that is, the number of infections. We are given that

$$\frac{dP}{dt} = I(t) = 16 + 242t \qquad P(0) = 80$$

This is an initial-value problem for a pure-time differential equation. The general solution is given by the antiderivative of dP/dt:

$$P(t) = 16t + 121t^2 + C$$

Then $P(0) = C$, but we are given that $P(0) = 80$, so $C = 80$. The solution is

$$P(t) = 80 + 16t + 121t^2$$

The projected number of infections in 1990 is

$$P(8) = 80 + 16 \cdot 8 + 121 \cdot 8^2 = 7952$$

The actual number was estimated to be 7200. ■

EXAMPLE 5 | Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

SOLUTION The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

we have $C = -3$. Therefore the required function is

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$ ■

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$. This means that the position function is an antiderivative of the velocity function. Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration. If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

EXAMPLE 6 | A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

SOLUTION Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$ ■

EXERCISES 4.6

1–20 Find the most general antiderivative of the function. (Check your answer by differentiation.)

1. $f(x) = \frac{1}{2} + \frac{3}{4}x^2 - \frac{4}{5}x^3$

2. $f(x) = 8x^9 - 3x^6 + 12x^3$

3. $f(x) = (x + 1)(2x - 1)$

4. $f(x) = x(2 - x)^2$

5. $f(x) = 5x^{1/4} - 7x^{3/4}$

6. $f(x) = 2x + 3x^{1.7}$

7. $f(x) = 6\sqrt{x} - \sqrt[6]{x}$

8. $f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4}$

9. $f(x) = \sqrt{2}$

10. $f(x) = e^2$

11. $c(t) = \dfrac{3}{t^2}, \quad t > 0$

12. $h(m) = \dfrac{2}{\sqrt{m}}$

13. $g(\theta) = \cos\theta - 5\sin\theta$

14. $f(x) = 2\sqrt{x} + 6\cos x$

15. $v(s) = 4s + 3e^s$

16. $u(r) = e^{-2r}$

17. $f(u) = \dfrac{u^4 + 3\sqrt{u}}{u^2}$

18. $f(x) = 3e^x + 7\sec^2 x$

19. $f(t) = \dfrac{t^4 - t^2 + 1}{t^2}$

20. $f(x) = \dfrac{1 + x - x^2}{x}$

21–28 Solve the initial-value problem.

21. $\dfrac{dy}{dt} = t^2 + 1, \quad t \geqslant 0, \quad y = 6$ when $t = 0$

22. $\dfrac{dy}{dt} = 1 + \dfrac{2}{t}, \quad t > 0, \quad y = 5$ when $t = 1$

23. $\dfrac{dP}{dt} = 2e^{3t}, \quad t \geqslant 0, \quad P(0) = 1$

24. $\dfrac{dm}{dt} = 100e^{-0.4t}, \quad t \geqslant 0, \quad m(0) = 50$

25. $\dfrac{dr}{d\theta} = \cos\theta + \sec\theta\tan\theta, \quad 0 < \theta < \pi/2, \quad r(\pi/3) = 4$

26. $\dfrac{dy}{dx} = x^2 + 1 + \dfrac{1}{x^2 + 1}, \quad y = 0$ when $x = 1$

27. $\dfrac{du}{dt} = \sqrt{t} + \dfrac{2}{\sqrt{t}}, \quad t > 0, \quad u(1) = 5$

28. $\dfrac{dv}{dt} = e^{-t}(1 + e^{2t}), \quad t \geqslant 0, \quad v(0) = 3$

29–40 Find f.

29. $f''(x) = 6x + 12x^2$

30. $f''(x) = 2 + x^3 + x^6$

31. $f''(x) = \frac{2}{3}x^{2/3}$

32. $f''(x) = 6x + \sin x$

33. $f'(x) = 1 - 6x, \quad f(0) = 8$

34. $f'(x) = 8x^3 + 12x + 3, \quad f(1) = 6$

35. $f'(x) = \sqrt{x}(6 + 5x), \quad f(1) = 10$

36. $f'(x) = 2x - 3/x^4, \quad x > 0, \quad f(1) = 3$

37. $f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$

38. $f''(x) = 8x^3 + 5, \quad f(1) = 0, \quad f'(1) = 8$

39. $f''(x) = 2 - 12x, \quad f(0) = 9, \quad f(2) = 15$

40. $f''(t) = 2e^t + 3\sin t, \quad f(0) = 0, \quad f(\pi) = 0$

41. Bacteria culture A culture of the bacterium *Rhodobacter sphaeroides* initially has 25 bacteria and t hours later increases at a rate of $3.4657e^{0.1386t}$ bacteria per hour. Find the population size after four hours.

42. A sample of cesium-37 with an initial mass of 75 mg decays t years later at a rate of $1.7325e^{-0.0231t}$ mg/year. Find the mass of the sample after 20 years.

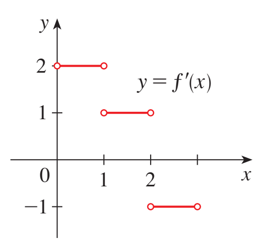
43. A particle moves along a straight line with velocity function $v(t) = \sin t - \cos t$ and its initial displacement is $s(0) = 0$ m. Find its position function $s(t)$.

44. A particle moves with acceleration function $a(t) = 5 + 4t - 2t^2$. Its initial velocity is $v(0) = 3$ m/s and its initial displacement is $s(0) = 10$ m. Find its position after t seconds.

45. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
(a) Find the distance of the stone above ground level at time t. Use the fact that acceleration due to gravity is $g \approx 9.8$ m/s^2.
(b) How long does it take the stone to reach the ground?
(c) With what velocity does it strike the ground?

46. To prove Theorem 1, let F and G be any two antiderivatives of f on I and let $H = G - F$.
(a) If x_1 and x_2 are any two numbers in I with $x_1 < x_2$, apply the Mean Value Theorem on the interval $[x_1, x_2]$ to show that $H(x_1) = H(x_2)$. Why does this show that H is a constant function?
(b) Deduce Theorem 1 from the result of part (a).

47. The graph of f' is shown in the figure. Sketch the graph of f if f is continuous on $[0, 3]$ and $f(0) = -1$.



48. Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

Chapter 4 REVIEW

CONCEPT CHECK

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

2. (a) What does the Extreme Value Theorem say?
(b) Explain how the Closed Interval Method works.

3. (a) State Fermat's Theorem.
(b) Define a critical number of f.

4. State the Mean Value Theorem and give a geometric interpretation.

5. (a) State the Increasing/Decreasing Test.
(b) What does it mean to say that f is concave upward on an interval I?
(c) State the Concavity Test.
(d) What are inflection points? How do you find them?

6. (a) State the First Derivative Test.
(b) State the Second Derivative Test.
(c) What are the relative advantages and disadvantages of these tests?

7. (a) What does l'Hospital's Rule say?
(b) How can you use l'Hospital's Rule if you have a product $f(x)\,g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$?
(c) How can you use l'Hospital's Rule if you have a difference $f(x) - g(x)$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$?

8. (a) What is an equilibrium of the recursive sequence $x_{t+1} = f(x_t)$?
(b) What is a stable equilibrium? An unstable equilibrium?
(c) State the Stability Criterion.

9. (a) What is an antiderivative of a function f?
(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

Answers to the Concept Check can be found on the back endpapers.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $f'(c) = 0$, then f has a local maximum or minimum at c.

2. If f has an absolute minimum value at c, then $f'(c) = 0$.

3. If f is continuous on (a, b), then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).

4. If f is differentiable and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.

5. If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1, 6)$.

6. If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of the curve $y = f(x)$.

7. If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.

8. There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.

9. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

10. There exists a function f such that $f(x) < 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

11. If f and g are increasing on an interval I, then $f + g$ is increasing on I.

12. If f and g are increasing on an interval I, then $f - g$ is increasing on I.

13. If f and g are increasing on an interval I, then fg is increasing on I.

14. If f and g are positive increasing functions on an interval I, then fg is increasing on I.

15. If f is increasing and $f(x) > 0$ on I, then $g(x) = 1/f(x)$ is decreasing on I.

16. If f is even, then f' is even.

17. If f is periodic, then f' is periodic.

18. $\displaystyle\lim_{x \to 0} \frac{x}{e^x} = 1$