the second derivative f'' in Figure 21. We see that f'' changes from positive to negative when $x \approx -1.23$ and from negative to positive when $x \approx 0.19$. So, correct to two decimal places, f is concave upward on $(-\infty, -1.23)$ and $(0.19, \infty)$ and concave downward on $(-1.23, 0.19)$. The inflection points are $(-1.23, -10.18)$ and $(0.19, -0.05)$.

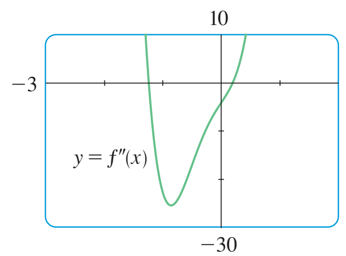
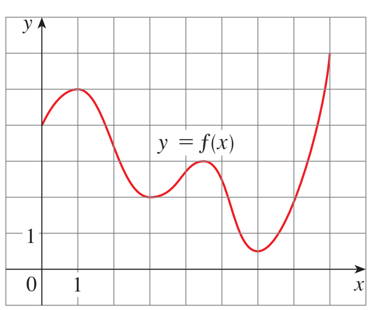


FIGURE 21

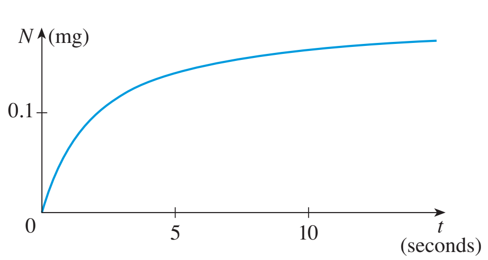
We have discovered that no single graph reveals all the important features of this polynomial. But Figures 18 and 20, when taken together, do provide an accurate picture. ■

EXERCISES 4.2

1. Use the graph of f to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[0, 8]$.



2. Foraging Many animals forage on resources that are distributed in discrete patches. For example, bumblebees visit many flowers, foraging on nectar from each. The amount of nectar $N(t)$ consumed from any flower increases with the amount of time spent at that flower, but with diminishing returns, as illustrated.



(a) What does this mean about the first and second derivatives of N?

(b) What is the average rate at which nectar is consumed over the first 10 seconds?

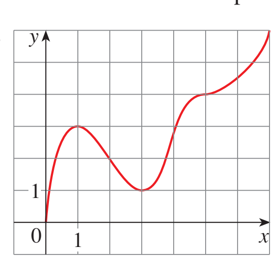
(c) The Mean Value Theorem tells us that there exists a time at which the instantaneous rate of nectar consumption is equal to the average found in part (b). Illustrate this idea graphically and estimate the time at which this occurs.

3. Suppose that $3 \leqslant f'(x) \leqslant 5$ for all values of x. Show that $18 \leqslant f(8) - f(2) \leqslant 30$.

4–5 Use the given graph of f to find the following.

(a) The open intervals on which f is increasing.
(b) The open intervals on which f is decreasing.
(c) The open intervals on which f is concave upward.
(d) The open intervals on which f is concave downward.
(e) The coordinates of the points of inflection.

4.



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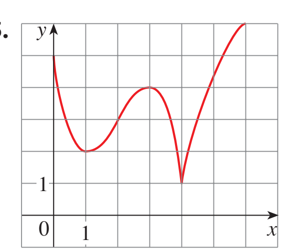


6. Suppose you are given a formula for a function f.

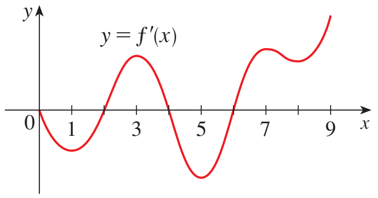
(a) How do you determine where f is increasing or decreasing?

(b) How do you determine where the graph of f is concave upward or concave downward?

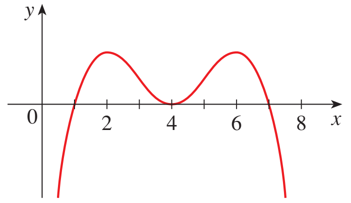
(c) How do you locate inflection points?

7. (a) State the First Derivative Test.
(b) State the Second Derivative Test. Under what circumstances is it inconclusive? What do you do if it fails?

8. The graph of the first derivative f' of a function f is shown.
(a) On what intervals is f increasing? Explain.
(b) At what values of x does f have a local maximum or minimum? Explain.
(c) On what intervals is f concave upward or concave downward? Explain.
(d) What are the x-coordinates of the inflection points of f? Why?



9. In each part state the x-coordinates of the inflection points of f. Give reasons for your answers.
(a) The curve is the graph of f.
(b) The curve is the graph of f'.
(c) The curve is the graph of f''.



10. HIV prevalence The table gives the number of HIV-infected men in San Francisco from 1982 to 1991.

Year	Number of infections	Year	Number of infections
1982	80	1987	3500
1983	300	1988	4500
1984	700	1989	6000
1985	1500	1990	7200
1986	2500	1991	9000

(a) If $H(t)$ is the number of infected men at time t, plot the values of $H(t)$. What does the direction of concavity appear to be? Provide a biological interpretation.
(b) Use the table to construct a table of estimated values for $H'(t)$.
(c) Use the table of values of $H'(t)$ in part (b) to construct a table of values for $H''(t)$. Do the values corroborate your answer to part (a)?

11–20
(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

11. $f(x) = 2x^3 + 3x^2 - 36x$

12. $f(x) = 4x^3 + 3x^2 - 6x + 1$

13. $f(x) = x^4 - 2x^2 + 3$

14. $f(x) = \dfrac{x^2}{x^2 + 3}$

15. $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

16. $f(x) = \cos^2 x - 2\sin x, \quad 0 \leqslant x \leqslant 2\pi$

17. $f(x) = e^{2x} + e^{-x}$ **18.** $f(x) = x^2 \ln x$

19. $f(x) = (\ln x)/\sqrt{x}$ **20.** $f(x) = \sqrt{x}\, e^{-x}$

21–22 Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

21. $f(x) = x + \sqrt{1 - x}$ **22.** $f(x) = \dfrac{x}{x^2 + 4}$

23. Suppose f'' is continuous on $(-\infty, \infty)$.
(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?
(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

24. (a) Find the critical numbers of $f(x) = x^4(x - 1)^3$.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
(c) What does the First Derivative Test tell you?

25–36
(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a)–(c) to sketch the graph. Check your work with a graphing device if you have one.

25. $f(x) = 2x^3 - 3x^2 - 12x$ **26.** $f(x) = 2 + 3x - x^3$

27. $f(x) = 2 + 2x^2 - x^4$ **28.** $g(x) = 200 + 8x^3 + x^4$

29. $h(x) = (x + 1)^5 - 5x - 2$ **30.** $h(x) = x^5 - 2x^3 + x$

31. $A(x) = x\sqrt{x + 3}$ **32.** $B(x) = 3x^{2/3} - x$

33. $C(x) = x^{1/3}(x + 4)$ **34.** $f(x) = \ln(x^4 + 27)$

35. $f(\theta) = 2\cos\theta + \cos^2\theta, \quad 0 \leqslant \theta \leqslant 2\pi$

36. $f(t) = t + \cos t, \quad -2\pi \leqslant t \leqslant 2\pi$

37–44
(a) Find the vertical and horizontal asymptotes.
(b) Find the intervals of increase or decrease.
(c) Find the local maximum and minimum values.
(d) Find the intervals of concavity and the inflection points.
(e) Use the information from parts (a)–(d) to sketch the graph of f.

37. $f(x) = \dfrac{x^2}{x^2 - 1}$ **38.** $f(x) = \dfrac{x^2}{(x-2)^2}$

39. $f(x) = \sqrt{x^2 + 1} - x$

40. $f(x) = x \tan x, \quad -\pi/2 < x < \pi/2$

41. $f(x) = \ln(1 - \ln x)$ **42.** $f(x) = \dfrac{e^x}{1 + e^x}$

43. $f(x) = e^{-1/(x+1)}$ **44.** $f(x) = e^{\arctan x}$

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45. Suppose the derivative of a function f is $f'(x) = (x+1)^2(x-3)^5(x-6)^4$. On what interval is f increasing?

46. Use the methods of this section to sketch the curve $y = x^3 - 3a^2x + 2a^3$, where a is a positive constant. What do the members of this family of curves have in common? How do they differ from each other?

47. Let $f(t)$ be the temperature at time t where you live and suppose that at time $t = 3$ you feel uncomfortably hot. How do you feel about the given data in each case?
(a) $f'(3) = 2, \quad f''(3) = 4$
(b) $f'(3) = 2, \quad f''(3) = -4$
(c) $f'(3) = -2, \quad f''(3) = 4$
(d) $f'(3) = -2, \quad f''(3) = -4$

48. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.
(a) Sketch a possible graph for f.
(b) How many solutions does the equation $f(x) = 0$ have? Why?
(c) Is it possible that $f'(2) = \frac{1}{3}$? Why?

49. Coffee is being poured into the mug shown in the figure at a constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?

50. Antibiotic pharmacokinetics Suppose that antibiotics are injected into a patient to treat a sinus infection. The antibiotics circulate in the blood, slowly diffusing into the sinus cavity while simultaneously being filtered out of the blood by the liver. In Chapter 10 we will derive a model for the concentration of the antibiotic in the sinus cavity as a function of time since the injection:

$$c(t) = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} \quad \text{where } \beta > \alpha > 0$$

(a) At what time does c have its maximum value?
(b) At what time does the inflection point occur? What is the significance of the inflection point for the concentration function?
(c) Sketch the graph of c.

51. A **drug-loading curve** describes the level of medication in the bloodstream after a drug is administered. A *surge function* $S(t) = At^pe^{-kt}$ is often used to model the loading curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

52. Mutation accumulation When a population is subjected to a mutagen, the fraction of the population that contains at least one mutation increases with the duration of the exposure. A commonly used equation describing this fraction is $f(t) = 1 - e^{-\mu t}$, where μ is the mutation rate and is positive. Suppose we have two populations, A and B. Population A is subjected to the mutagen for 3 min whereas, with population B, half of the individuals are subjected to the mutagen for 2 min and the other half for 4 min. Which population will have the largest fraction of mutants? Explain your answer using derivatives.

53. A **dose response curve** in pharmacology is a plot of the effectiveness R of a drug as a function of the drug concentration c. Such curves typically increase with an S-shape, a simple mathematical model being

$$R(c) = \frac{c^2}{3 + c^2}$$

(a) At what drug concentration does the inflection point occur?
(b) Suppose we have two different treatment protocols, one where the concentration is held steady at 2 and another in which the concentration varies through time, spending equal amounts of time at 1.5 and 2.5. Which protocol would have the greater response?

54. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

55. In the theory of relativity, the mass of a particle is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v relative to the observer, and c is the speed of light. Sketch the graph of m as a function of v.

56. In the theory of relativity, the energy of a particle is

$$E = \sqrt{m_0^2c^4 + h^2c^2/\lambda^2}$$

where m_0 is the rest mass of the particle, λ is its wave length, and h is Planck's constant. Sketch the graph of E as a function of λ. What does the graph say about the energy?

57. Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

58. For what values of the numbers a and b does the function

$$f(x) = axe^{bx^2}$$

have the maximum value $f(2) = 1$?

59–62 Produce graphs of f that reveal all the important aspects of the curve. In particular, you should use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

59. $f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29$

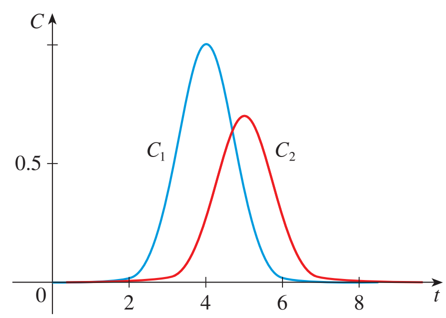
60. $f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x$

61. $f(x) = x^2 - 4x + 7\cos x, \quad -4 \leqslant x \leqslant 4$

62. $f(x) = \tan x + 5\cos x$

63. Growth rate A 20-year-old university student weighs 138 lb and had a birth weight of 6 lb. Prove that at some point in her life she was growing at a rate of 6.6 pounds per year.

64. Antibiotic concentration Suppose an antibiotic is administered orally. It is first absorbed into the bloodstream, from which it passes into the sinus cavity. It is also metabolized from both sites. The concentrations $C_1(t)$ in the blood and $C_2(t)$ in the sinus cavity are shown. Prove that there is a time when the concentration in each site is increasing at the same rate. Prove that there is also a time when the concentration in each site is decreasing at the same rate.



65. Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three x-intercepts x_1, x_2, and x_3, show that the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/3$.

66. For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

4.3 L'Hospital's Rule: Comparing Rates of Growth

Indeterminate Quotients

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves *near* 1. In