Chapter 3 REVIEW

CONCEPT CHECK

1. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

2. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

3. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.
   (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$.
   (b) The instantaneous rate of change of y with respect to x at $x = x_1$.

4. Define the second derivative of f. If $f(t)$ is the position function of a particle, how can you interpret the second derivative?

5. (a) What does it mean for f to be differentiable at a?
   (b) What is the relation between the differentiability and continuity of a function?
   (c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.

6. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

7. State each differentiation rule both in symbols and in words.
   (a) The Power Rule (b) The Constant Multiple Rule
   (c) The Sum Rule (d) The Difference Rule
   (e) The Product Rule (f) The Quotient Rule
   (g) The Chain Rule

8. State the derivative of each function.
   (a) $y = x^n$ (b) $y = e^x$ (c) $y = b^x$
   (d) $y = \ln x$ (e) $y = \log_b x$ (f) $y = \sin x$
   (g) $y = \cos x$ (h) $y = \tan x$ (i) $y = \csc x$
   (j) $y = \sec x$ (k) $y = \cot x$ (l) $y = \tan^{-1} x$

9. (a) How is the number e defined?
   (b) Express e as a limit.
   (c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = b^x$?
   (d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_b x$?

10. (a) Explain how implicit differentiation works. When should you use it?
    (b) Explain how logarithmic differentiation works. When should you use it?

11. Write an expression for the linearization of f at a.

12. Write an expression for the nth-degree Taylor polynomial of f centered at a.

Answers to the Concept Check can be found on the back endpapers.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is continuous at a, then f is differentiable at a.

2. If $f'(r)$ exists, then $\lim_{x \to r} f(x) = f(r)$.

3. If f and g are differentiable, then
$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

4. If f and g are differentiable, then
$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$$

5. If f and g are differentiable, then
$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

6. If f is differentiable, then $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

7. If f is differentiable, then $\dfrac{d}{dx}f\left(\sqrt{x}\right) = \dfrac{f'(x)}{2\sqrt{x}}$.

8. If $y = e^2$, then $y' = 2e$.

9. $\dfrac{d}{dx}(10^x) = x10^{x-1}$

10. $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$

11. $\dfrac{d}{dx}(\tan^2 x) = \dfrac{d}{dx}(\sec^2 x)$

12. $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

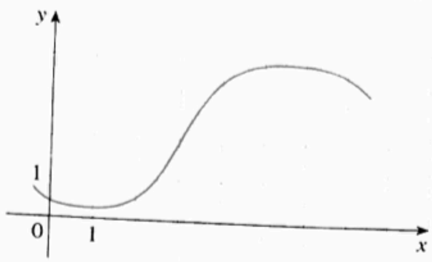
13. If $g(x) = x^5$, then $\displaystyle\lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = 80$

14. An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

EXERCISES

1. For the function f whose graph is shown, arrange the following numbers in increasing order:

$0 \quad 1 \quad f'(2) \quad f'(3) \quad f'(5) \quad f''(5)$



2. **Life expectancy** The table shows how the life expectancy $L(t)$ in Bangladesh has changed from 1990 to 2010.

t	1990	1995	2000	2005	2010
$L(t)$	56	61	65	68	71

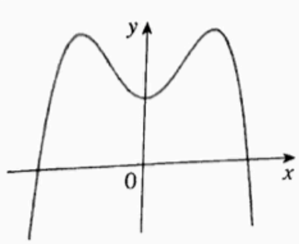
(a) Calculate the average rate of change of the life expectancy $L(t)$ with respect to time over the following time intervals.
(i) [1990, 2000] (ii) [1995, 2000]
(iii) [2000, 2010] (iv) [2000, 2005]
(b) Estimate the value of $L'(2000)$.

3. The total cost of repaying a student loan at an interest rate of r% per year is $C = f(r)$.
(a) What is the meaning of the derivative $f'(r)$? What are its units?
(b) What does the statement $f'(10) = 1200$ mean?
(c) Is $f'(r)$ always positive or does it change sign?

4. (a) Use the definition of a derivative to find $f'(2)$, where $f(x) = x^3 - 2x$.
(b) Find an equation of the tangent line to the curve $y = x^3 - 2x$ at the point (2, 4).
(c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.

5–7 Trace or copy the graph of the function. Then sketch a graph of its derivative directly beneath.

5.



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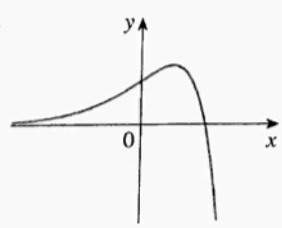

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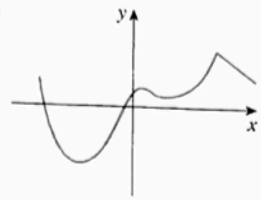
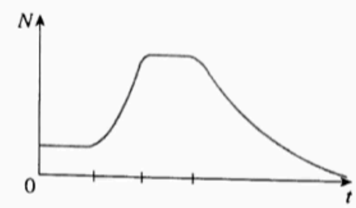
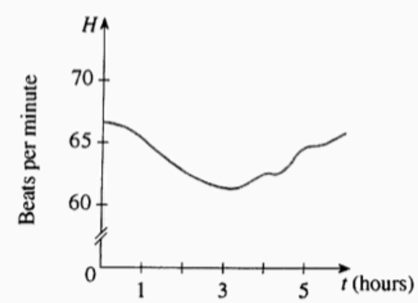


8. **Bacteria count** Shown is a typical graph of the number N of bacteria grown in a bacteria culture as a function of time t.
(a) What is the meaning of the derivative $N'(t)$?
(b) Sketch the graph of $N'(t)$.



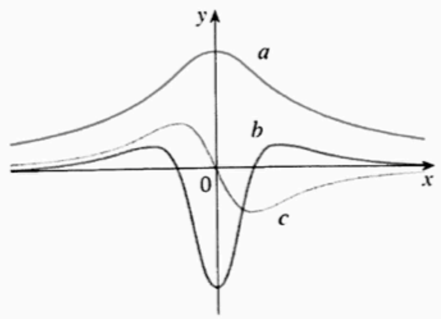
9. **Antihypertension medication** The figure shows the heart rate $H(t)$ after a patient has taken nifedipine tablets.
(a) What is the meaning of the derivative $H'(t)$?
(b) Sketch the graph of $H'(t)$.



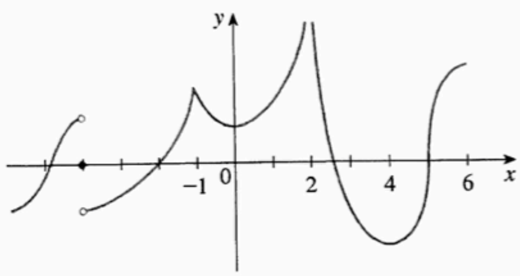
10. (a) Find the asymptotes of the graph of $f(x) = \dfrac{4 - x}{3 + x}$ and use them to sketch the graph.
(b) Use your graph from part (a) to sketch the graph of f'.
(c) Use the definition of a derivative to find $f'(x)$.
(d) Use a calculator to graph f' and compare with your sketch in part (b).

11. (a) If $f(x) = \sqrt{3 - 5x}$, use the definition of a derivative to find $f'(x)$.
(b) Find the domains of f and f'.
(c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.

12. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



13. The graph of f is shown. State, with reasons, the numbers at which f is not differentiable.

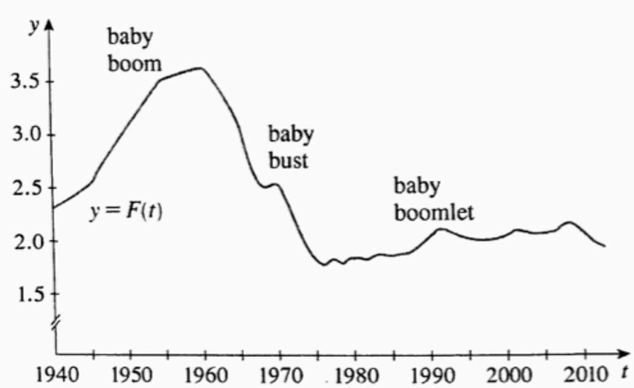


14. The **total fertility rate** at time t, denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in the United States shows the fluctuations from 1940 to 2010.

(a) Estimate the values of $F'(1950)$, $F'(1965)$, and $F'(1987)$.

(b) What are the meanings of these derivatives?

(c) Can you suggest reasons for the values of these derivatives?



15–50 Calculate y'.

15. $y = (x^4 - 3x^2 + 5)^3$

16. $y = \cos(\tan x)$

17. $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$

18. $y = \dfrac{3x - 2}{\sqrt{2x + 1}}$

19. $y = 2x\sqrt{x^2 + 1}$

20. $y = \dfrac{e^x}{1 + x^2}$

21. $y = e^{\sin 2\theta}$

22. $y = e^{-t}(t^2 - 2t + 2)$

23. $y = \dfrac{t}{1 - t^2}$

24. $y = e^{mx}\cos nx$

25. $y = \dfrac{e^{1/x}}{x^2}$

26. $y = \left(\dfrac{u - 1}{u^2 + u + 1}\right)^4$

27. $xy^4 + x^2y = x + 3y$

28. $y = \ln(\csc 5x)$

29. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta}$

30. $x^2 \cos y + \sin 2y = xy$

31. $y = e^{cx}(c \sin x - \cos x)$

32. $y = \ln(x^2e^x)$

33. $y = \log_5(1 + 2x)$

34. $y = (\ln x)^{\cos x}$

35. $\sin(xy) = x^2 - y$

36. $y = \sqrt{t \ln(t^4)}$

37. $y = 3^{x \ln x}$

38. $xe^y = y - 1$

39. $y = \ln \sin x - \frac{1}{2}\sin^2 x$

40. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5}$

41. $y = x \tan^{-1}(4x)$

42. $y = e^{\cos x} + \cos(e^x)$

43. $y = \ln|\sec 5x + \tan 5x|$

44. $y = 10^{\tan \pi\theta}$

45. $y = \tan^2(\sin \theta)$

46. $y = \ln\left|\dfrac{x^2 - 4}{2x + 5}\right|$

47. $y = \sin\left(\tan\sqrt{1 + x^3}\right)$

48. $y = \arctan\left(\arcsin\sqrt{x}\right)$

49. $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

50. $y = \sin^2\left(\cos\sqrt{\sin \pi x}\right)$

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51. If $f(t) = \sqrt{4t + 1}$, find $f''(2)$.

52. If $g(\theta) = \theta \sin\theta$, find $g''(\pi/6)$.

53. If $f(x) = 2^x$, find $f^{(n)}(x)$.

54. Find $f^{(n)}(x)$ if $f(x) = 1/(2 - x)$.

55–56 Find an equation of the tangent to the curve at the given point.

55. $y = 4\sin^2 x$, $(\pi/6, 1)$

56. $y = \dfrac{x^2 - 1}{x^2 + 1}$, $(0, -1)$

57–58 Find equations of the tangent line and normal line to the curve at the given point.

57. $y = (2 + x)e^{-x}$, $(0, 2)$

58. $x^2 + 4xy + y^2 = 13$, $(2, 1)$

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59. (a) If $f(x) = x\sqrt{5 - x}$, find $f'(x)$.

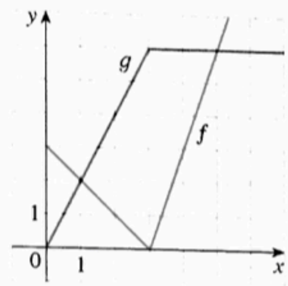
(b) Find equations of the tangent lines to the curve $y = x\sqrt{5 - x}$ at the points $(1, 2)$ and $(4, 4)$.

(c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.
(d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

60. (a) Graph the function $f(x) = x - 2\sin x$ in the viewing rectangle $[0, 8]$ by $[-2, 8]$.
(b) On which interval is the average rate of change larger: $[1, 2]$ or $[2, 3]$?
(c) At which value of x is the instantaneous rate of change larger: $x = 2$ or $x = 5$?
(d) Check your visual estimates in part (c) by computing $f'(x)$ and comparing the numerical values of $f'(2)$ and $f'(5)$.

61. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(g(x))$, where $f(2) = 3$, $g(2) = 5$, $g'(2) = 4$, $f'(2) = -2$, and $f'(5) = 11$. Find (a) $h'(2)$ and (b) $F'(2)$.

62. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



63–70 Find f' in terms of g'.

63. $f(x) = x^2 g(x)$

64. $f(x) = g(x^2)$

65. $f(x) = [g(x)]^2$

66. $f(x) = g(g(x))$

67. $f(x) = g(e^x)$

68. $f(x) = e^{g(x)}$

69. $f(x) = \ln |g(x)|$

70. $f(x) = g(\ln x)$

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71. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

72. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

73. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

74. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

75. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

76. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2t^2}$, $t \geq 0$, where b and c are positive constants.
(a) Find the velocity and acceleration functions.
(b) Show that the particle always moves in the positive direction.

77. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.
(a) Find the rate of change of the volume with respect to the height if the radius is constant.
(b) Find the rate of change of the volume with respect to the radius if the height is constant.

78. The **Michaelis-Menten equation** for the enzyme pepsin is

$$v = \frac{0.50[S]}{3.0 \times 10^{-4} + [S]}$$

where v is the rate of an enzymatic reaction and $[S]$ is the concentration of a substrate S. Calculate $dv/d[S]$ and interpret it.

79. Health care expenditures The US health care expenditures for 1970–2008 have been modeled by the function

$$E(t) = 101.35e^{0.088128t}$$

where t is the number of years elapsed since 1970 and E is measured in billions of dollars. According to this model, at what rate were health care expenditures increasing in 1980? In 2000?

80. Drug concentration The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t \to \infty} C(t) = 0$.
(b) Find $C'(t)$, the rate of change of drug concentration in the blood.
(c) When is this rate equal to 0?

81. Bacteria growth A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000?

82. Cobalt-60 has a half-life of 5.24 years.
(a) Find the mass that remains from a 100-mg sample after 20 years.
(b) How long would it take for the mass to decay to 1 mg?

83. Drug elimination Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = -kC(t)$, where k is a positive number called the *elimination constant* of the drug.

(a) If C_0 is the concentration at time $t = 0$, find the concentration at time t.

(b) If the body eliminates half the drug in 30 hours, how long does it take to eliminate 90% of the drug?

84. A cup of hot chocolate has temperature 80°C in a room kept at 20°C. After half an hour the hot chocolate cools to 60°C.

(a) What is the temperature of the chocolate after another half hour?

(b) When will the chocolate have cooled to 40°C?

85. The volume of a cube is increasing at a rate of 10 cm³/min. How fast is the surface area increasing when the length of an edge is 30 cm?

86. Yeast population The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

87. Use Newton's method to find the root of the equation $x^5 - x^4 + 3x^2 - 3x - 2 = 0$ in the interval $[1, 2]$ correct to six decimal places.

88. Use Newton's method to find all roots of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places.

89. (a) Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.

(b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.

90. (a) Find the first three Taylor polynomials for $f(x) = 4(x - 2)^{-2}$ centered at 0.

(b) Graph f and the Taylor polynomials from part (a) on the interval $[-1, 1]$ and comment on how well the polynomials approximate f.

91. Dialysis The project on page 458 models the removal of urea from the bloodstream via dialysis. In certain situations the duration of dialysis required, given that the initial urea concentration is c, where $c > 1$, is given by the equation

$$t = \ln\left(\frac{3c + \sqrt{9c^2 - 8c}}{2}\right)$$

(a) Use a linear approximation to estimate the time required if the initial concentration is near $c = 1$.

(b) Use a second-order Taylor polynomial to give a more accurate approximation.

92. Infectious disease outbreak size We have worked with the model

$$pe^{-qA} = 1 - A$$

where A is the fraction of the population infected, q is a measure of disease transmissibility, and p is the fraction of the population that is initially susceptible to infection.

(a) Use implicit differentiation to find the linear approximation of A as a function of q at $q = 0$.

(b) Find the second-order Taylor polynomial approximation for $A(q)$ at $q = 0$.

93. Express the limit

$$\lim_{\theta \to \pi/3} \frac{\cos\theta - 0.5}{\theta - \pi/3}$$

as a derivative and thus evaluate it.

94. Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x-axis and the tangent lines at P and Q is an equilateral triangle.