EXERCISES 3.7

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_b x$.

2–20 Differentiate the function.

2. $f(x) = x \ln x - x$

3. $f(x) = \sin(\ln x)$

4. $f(x) = \ln(\sin^2 x)$

5. $f(x) = \log_2(1 - 3x)$

6. $f(x) = \log_5(xe^x)$

7. $f(x) = \sqrt[5]{\ln x}$

8. $f(x) = \ln \sqrt[5]{x}$

10-14

9. $f(x) = \sin x \ln(5x)$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

11. $F(t) = \ln \dfrac{(2t + 1)^3}{(3t - 1)^4}$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right)$

14. $F(y) = y \ln(1 + e^y)$

15. $y = \ln|2 - x - 5x^2|$

16. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

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17. $y = \ln(e^{-x} + xe^{-x})$

18. $y = [\ln(1 + e^x)]^2$

19. $y = 2x \log_{10} \sqrt{x}$

20. $y = \log_2(e^{-x} \cos \pi x)$

21–22 Find y' and y''.

21-22

21. $y = x^2 \ln(2x)$

22. $y = \dfrac{\ln x}{x^2}$

23–24 Differentiate f and find the domain of f.

23. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

24. $f(x) = \ln \ln \ln x$

25–26 Find an equation of the tangent line to the curve at the given point.

25-26.

25. $y = \ln(x^2 - 3x + 1)$, $(3, 0)$

26. $y = x^2 \ln x$, $(1, 0)$

27. If $f(x) = \dfrac{\ln x}{x^2}$, find $f'(1)$.

28. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

29. Dialysis The project on page 458 models the removal of urea from the bloodstream via dialysis. Given that the initial urea concentration, measured in mg/mL, is c (where $c > 1$), the duration of dialysis required for certain conditions is given by the equation

$$t = \ln\left(\frac{3c + \sqrt{9c^2 - 8c}}{2}\right)$$

Calculate the derivative of t with respect to c and interpret it.

30. Genetic drift A population of fruit flies contains two genetically determined kinds of individuals: white-eyed flies and red-eyed flies. Suppose that a scientist maintains the population at constant size N by randomly choosing N juvenile flies after reproduction to form the next generation. Eventually, because of the random sampling in each generation, by chance the population will contain only a single type of fly. This is called *genetic drift*. Suppose that the initial fraction of the population that are white-eyed is p_0. An equation for the average number of generations required before all flies are white-eyed (given that this occurs instead of all flies being red-eyed) is

$$g = -2N \frac{1 - p_0}{p_0} \ln(1 - p_0)$$

Calculate the derivative of g with repect to p_0 and explain its meaning.

31. Carbon dating If N is the measured amount of ^{14}C in a fossil organism and N_0 is the amount in living organisms, then the estimated age of the fossil is given by the equation

$$a = \frac{5370}{\ln 2} \ln\left(\frac{N_0}{N}\right)$$

Calculate da/dN and interpret it.

32. Let $f(x) = \log_b(3x^2 - 2)$. For what value of b is $f'(1) = 3$?

33–41 Use logarithmic differentiation to find the derivative of the function.

33. $y = (2x + 1)^5(x^4 - 3)^6$

34. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$

35. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

36. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

37. $y = x^x$

38. $y = x^{\cos x}$

39. $y = (\cos x)^x$ **40.** $y = \sqrt{x}^x$

41. $y = (\tan x)^{1/x}$

42. Predator-prey dynamics In Chapter 7 we study a model for the population dynamics of a predator and its prey species. If $u(t)$ and $v(t)$ denote the prey and predator population sizes at time t, an equation relating the two is

$$ve^{-v}u^{\alpha}e^{-\alpha u} = c$$

where c and α are positive constants. Use logarithmic differentiation to obtain an equation relating the relative (per capita) rate of change of predator (that is, v'/v) to that of prey (that is, u'/u).

43–48 Find the derivative of the function. Simplify where possible.

43. $y = (\tan^{-1}x)^2$ **44.** $y = \tan^{-1}(x^2)$

45. $y = \arctan(\cos\theta)$ **46.** $f(x) = x\ln(\arctan x)$

47. $y = \tan^{-1}\left(x - \sqrt{1+x^2}\right)$ **48.** $y = \arctan\sqrt{\dfrac{1-x}{1+x}}$

49–50 Find the limit.

49. $\lim\limits_{x\to\infty} \arctan(e^x)$ **50.** $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x)$

51. Find y' if $y = \ln(x^2 + y^2)$.

52. Find y' if $x^y = y^x$.

53. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x-1)$.

54. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

55. Use the definition of derivative to prove that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

56. Show that $\lim\limits_{n\to\infty}\left(1 + \dfrac{x}{n}\right)^n = e^x$ for any $x > 0$.

3.8 Linear Approximations and Taylor Polynomials

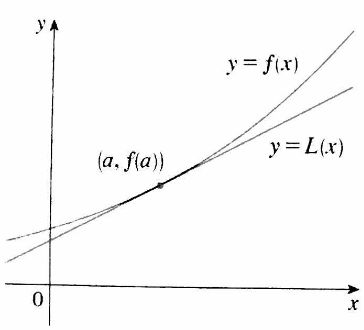

FIGURE 1

Tangent Line Approximations

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 3.1.5.) This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x-a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x-a) \tag{1}$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x-a) \tag{2}$$

is called the **linearization** of f at a.