where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly

$$\Delta y = f'(b)\,\Delta u + \varepsilon_2\,\Delta u = [f'(b) + \varepsilon_2]\,\Delta u \tag{10}$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. If we now substitute the expression for Δu from Equation 9 into Equation 10, we get

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\,\Delta x$$

so

$$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

As $\Delta x \to 0$, Equation 9 shows that $\Delta u \to 0$. So both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$
$$= f'(b)\,g'(a) = f'(g(a))\,g'(a)$$

This proves the Chain Rule. ■

EXERCISES 3.5

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

1. $y = \sqrt[3]{1 + 4x}$
2. $y = (2x^3 + 5)^4$
3. $y = \tan \pi x$
4. $y = \sin(\cot x)$
5. $y = e^{\sqrt{x}}$
6. $y = \sqrt{2 - e^x}$

7–36 Find the derivative of the function.

7. $F(x) = (x^4 + 3x^2 - 2)^5$
8. $F(x) = (4x - x^2)^{100}$
9. $F(x) = \sqrt{1 - 2x}$
10. $f(x) = (1 + x^4)^{2/3}$
11. $f(z) = \dfrac{1}{z^2 + 1}$
12. $f(t) = \sqrt[3]{1 + \tan t}$
13. $y = \cos(a^3 + x^3)$
14. $y = a^3 + \cos^3 x$
15. $h(t) = t^3 - 3^t$
16. $y = 3\cot(n\theta)$
17. $y = xe^{-kx}$
18. $y = e^{-2t}\cos 4t$
19. $y = (2x - 5)^4(8x^2 - 5)^{-3}$
20. $h(t) = (t^4 - 1)^3(t^3 + 1)^4$
21. $y = e^{x\cos x}$
22. $y = 10^{1-x^2}$
23. $y = \left(\dfrac{x^2 + 1}{x^2 - 1}\right)^3$
24. $G(y) = \left(\dfrac{y^2}{y + 1}\right)^5$
25. $y = \sec^2 x + \tan^2 x$
26. $y = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$
27. $y = \dfrac{r}{\sqrt{r^2 + 1}}$
28. $y = e^{k\tan\sqrt{x}}$
29. $y = \sin(\tan 2x)$
30. $f(t) = \sqrt{\dfrac{t}{t^2 + 4}}$
31. $y = 2^{\sin \pi x}$
32. $y = \sin(\sin(\sin x))$
33. $y = \cot^2(\sin\theta)$
34. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
35. $y = \cos\sqrt{\sin(\tan \pi x)}$
36. $y = 2^{3^{x^2}}$

37–40 Find y' and y''.

37. $y = \cos(x^2)$
38. $y = \cos^2 x$
39. $y = e^{\alpha x}\sin\beta x$
40. $y = e^{e^x}$

41–44 Find an equation of the tangent line to the curve at the given point.

41. $y = (1 + 2x)^{10}$, $(0, 1)$
42. $y = \sqrt{1 + x^3}$, $(2, 3)$
43. $y = \sin(\sin x)$, $(\pi, 0)$
44. $y = \sin x + \sin^2 x$, $(0, 0)$

45. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

46. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

47. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

48. Let f and g be the functions in Exercise 47.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = g(g(x))$, find $G'(3)$.

49. Suppose f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

50. Suppose f is differentiable on $\mathbb{R}$ and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

51. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

52. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

53. Find the 50th derivative of $y = \cos 2x$.

54. Find the 1000th derivative of $f(x) = xe^{-x}$.

55. The displacement of a particle on a vibrating string is given by the equation

$$s(t) = 10 + \tfrac{1}{4}\sin(10\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity and acceleration of the particle after t seconds.

56. Oral antibiotics In Example 1.3.7 we studied a model for antibiotic use in sinus infections. If x is the amount of antibiotic taken orally (in mg), then the function $h(x)$ gives the amount entering the bloodstream through the stomach. If x mg reaches the bloodstream, then $g(x)$ gives the amount that survives filtration by the liver. Finally, if x mg survives filtration by the liver, then $f(x)$ is absorbed into the sinus cavity. Thus, for a given dose x, the amount making it to the sinus cavity is $A(x) = f(g(h(x)))$. Suppose that a dose of 500 mg is given, $h(500) = 8$, $g(8) = 2$, $f(2) = 1.5$, and $h'(500) = 2.5$, $g'(8) = \frac{1}{4}$, and $f'(2) = 1$. Calculate $A'(x)$ and interpret your answer.

57. Gene regulation Genes produce molecules called mRNA that go on to produce proteins. High concentrations of protein inhibit the production of mRNA, leading to stable gene regulation. This process has been modeled (see Section 10.3) to show that the concentration of mRNA over time is given by the equation

$$m(t) = \tfrac{1}{2}e^{-t}(\sin t - \cos t) + \tfrac{1}{2}$$

What is the rate of change of mRNA concentration as a function of time?

58. World population growth In Example 1.4.3 we modeled the world population from 1900 to 2010 with the exponential function

$$P(t) = (1436.53) \cdot (1.01395)^t$$

where $t = 0$ corresponds to the year 1900 and $P(t)$ is measured in millions. According to this model, what was the rate of increase of world population in 1920? In 1950? In 2000?

59. Blood alcohol concentration In Section 3.1 we discussed an experiment in which the average BAC of eight male subjects was measured after consumption of 15 mL of ethanol (corresponding to one alcoholic drink). The resulting data were modeled by the concentration function

$$C(t) = 0.0225te^{-0.0467t}$$

where t is measured in minutes after consumption and C is measured in mg/mL.
(a) How rapidly was the BAC increasing after 10 minutes?
(b) How rapidly was it decreasing half an hour later?

60. Logistic growth in Japan The midyear population in Japan from 1960 to 2010 has been modeled by the function

$$P(t) = 94{,}000 + \frac{32{,}658.5}{1 + 12.75e^{-0.1706t}}$$

where t is measured in years since 1960 and $P(t)$ is measured in thousands. According to this model, how quickly was the Japanese population growing in 1970? In 1990?

61. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that has heard the rumor at time t and a and k are positive constants.
(a) Find $\lim_{t\to\infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

62. In Example 1.3.4 we arrived at a model for the length of daylight (in hours) in Philadelphia on the tth day of the year:

$$L(t) = 12 + 2.8\sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Use this model to compare how the number of hours of daylight is increasing in Philadelphia on March 21 and May 21.

63–64
(a) Find y' by implicit differentiation.
(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.
(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

63. $xy + 2x + 3x^2 = 4$

64. $\cos x + \sqrt{y} = 5$

65–76 Find dy/dx by implicit differentiation.

65. $x^3 + y^3 = 1$

66. $2\sqrt{x} + \sqrt{y} = 3$

67. $x^2 + xy - y^2 = 4$

68. $2x^3 + x^2y - xy^3 = 2$

69. $x^4(x + y) = y^2(3x - y)$

70. $y^5 + x^2y^3 = 1 + ye^{x^2}$

71. $4\cos x \sin y = 1$

72. $1 + x = \sin(xy^2)$

73. $e^{x/y} = x - y$

74. $\tan(x - y) = \dfrac{y}{1 + x^2}$

75. $e^y \cos x = 1 + \sin(xy)$

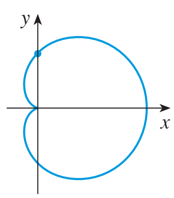
76. $\sin x + \cos y = \sin x \cos y$

77–80 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

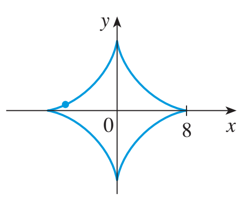
77. $x^2 + xy + y^2 = 3$, $(1, 1)$ (ellipse)

78. $x^2 + 2xy - y^2 + x = 2$, $(1, 2)$ (hyperbola)

79. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, $\left(0, \frac{1}{2}\right)$ (cardioid)



80. $x^{2/3} + y^{2/3} = 4$, $\left(-3\sqrt{3}, 1\right)$ (astroid)



81. Infectious disease outbreak size In Example 13 we used the equation

$$\rho e^{-qA} = 1 - A$$

to determine the rate of increase of the outbreak size A with respect to the transmissibility q. Use this same equation to find the rate of change of A with respect to ρ, the fraction of the population that is initially susceptible to infection.

82. The **logistic difference equation with migration** is of the form

$$N_{t+1} = N_t + N_t(1 - N_t) + m$$

where N_t is the population at time t and m is the migration rate. Suppose that as $t \to \infty$ the population size approaches a limiting value N.
(a) What equation does N satisfy?
(b) Use implicit differentiation to find the rate of change of N with respect to m.
(c) Find an explicit expression for N as a function of m, differentiate it, and compare with your answer to part (b).

83. If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

84. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.
(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

85. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

86. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

87. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

88. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this instant?

89. Bone mass In Example 1.1.6 we found an expression for the mass m of a human femur of length L in terms of the outer radius r, the inner radius r_{in}, and their ratio $k = r_{in}/r$. More generally, if the bone density is ρ, measured in g/cm^3, then bone mass is given by the equation

$$m = \pi r^2 L[\rho - (\rho - 1)k^2]$$

It may happen that both ρ and k change with age, t.
(a) If ρ changes during aging, find an expression for the rate of change of m with respect to t.
(b) If k changes during aging, find an expression for the rate of change of m with respect to t.

90. The **von Bertalanffy growth function**

$$L(t) = L_\infty - (L_\infty - L_0)e^{-kt}$$

where k is a positive constant, models the length L of a fish as a function of t, the age of the fish. Here L_0 is the length at birth and L_∞ is the final length. Suppose that the mass of a fish of length L is given by $M = aL^3$, where a is a positive constant. Calculate the rate of change of mass with respect to age.

91. Habitat fragmentation and species conservation The size of a class-structured population is modeled in Section 8.8. In certain situations the long-term growth rate of a population is given by $r = \frac{1}{2}\left(1 + \sqrt{1 + 8s}\right)$, where s is the annual survival probability of juveniles. Suppose this survival probability is related to habitat area a by a function $s(a)$. Determine an expression for the rate of change of growth rate with respect to a change in habitat area.

92. Blood flow In Example 3.3.9 we discussed Poiseuille's law of laminar flow:

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

where v is the blood velocity at a distance r from the center of a blood vessel (a vein or artery) in the shape of a tube with radius R and length l, P is the pressure difference between the ends of the tube, and η is the viscosity of the blood. In very cold weather, blood vessels in the hands and feet contract. Suppose that a blood vessel with $l = 1$ cm, $P = 1500$ dynes/cm^2, $\eta = 0.027$, and $R = 0.01$ cm contracts so that $R'(t) = -0.0005$ cm/min at a particular moment. Calculate dv/dt, the rate of change of the blood flow, at the center of the blood vessel at that time.

93. Angiotensin-converting enzyme (ACE) inhibitors are a type of blood pressure medication that reduces blood pressure by dilating blood vessels. Suppose that the radius R of a blood vessel is related to the dosage x of the medication by the function $R(x)$. One version of Poiseuille's Law gives the relationship between the blood pressure P and the radius as $P = 4\eta lv/R^2$, where v is the blood velocity at the center of the vessel, η is the viscosity of the blood, and l is the length of the blood vessel. Determine the rate of change of blood pressure with respect to dosage.

94. Brain size in fish Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain weight increasing when the average length was 18 cm?

95. Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta}(\sin\theta) = \frac{\pi}{180}\cos\theta$$

(This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus: The differentiation formulas would not be as simple if we used degree measure.)

96. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

3.6 Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k. Why is this reasonable? Suppose we have a population (of bacteria, for instance) with size $P = 1000$ and at a certain time it is growing at a rate of $P' = 300$ bacteria per hour. Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the new population was growing at a rate of 300 bacteria per hour. We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided there's enough room and nutrition). So if we double the size, we double the growth rate. In general, it seems reasonable that the growth rate should be proportional to the size.

Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what