When you memorize this table, it is helpful to notice that the minus signs go with the derivatives of the "cofunctions," that is, cosine, cosecant, and cotangent.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

EXAMPLE 7 | Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

SOLUTION The Quotient Rule gives

$$f'(x) = \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,(\tan x - 1)}{(1 + \tan x)^2}$$

In simplifying the answer we have used the identity $\tan^2 x + 1 = \sec^2 x$.

Since $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = n\pi + \pi/4$, where n is an integer (see Figure 6). ■

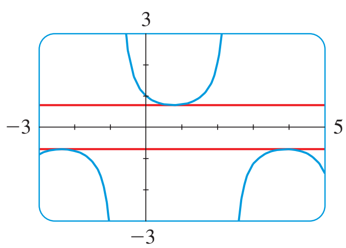

FIGURE 6
The horizontal tangents in Example 7

EXERCISES 3.4

1. Find the derivative of $f(x) = (1 + 2x^2)(x - x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function

$$F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

3–34 Differentiate.

3. $f(x) = (x^3 + 2x)e^x$
4. $g(x) = \sqrt{x}\, e^x$
5. $g(t) = t^3 \cos t$
6. $h(t) = e^t \sin t$
7. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3)$
8. $R(t) = (t + e^t)(3 - \sqrt{t})$
9. $f(x) = \sin x + \frac{1}{2}\cot x$
10. $y = 2\csc x + 5\cos x$
11. $h(\theta) = \theta \csc \theta - \cot \theta$
12. $y = u(a \cos u + b \cot u)$
13. $y = \dfrac{e^x}{x^2}$
14. $y = \dfrac{e^x}{1 + x}$
15. $g(x) = \dfrac{3x - 1}{2x + 1}$
16. $f(t) = \dfrac{2t}{4 + t^2}$
17. $y = \dfrac{x^3}{1 - x^2}$
18. $y = \dfrac{x + 1}{x^3 + x - 2}$
19. $y = \dfrac{t^2 + 2}{t^4 - 3t^2 + 1}$
20. $y = \dfrac{t}{(t - 1)^2}$

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21. $y = (r^2 - 2r)e^r$

22. $y = \dfrac{1}{s + ke^s}$

23. $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta}$

24. $y = \dfrac{1 + \sin x}{x + \cos x}$

25. $y = \dfrac{\sin x}{x^2}$

26. $y = \dfrac{1 - \sec x}{\tan x}$

27. $y = \dfrac{v^3 - 2v\sqrt{v}}{v}$

28. $z = w^{3/2}(w + ce^w)$

29. $f(t) = \dfrac{2t}{2 + \sqrt{t}}$

30. $g(t) = \dfrac{t - \sqrt{t}}{t^{1/3}}$

31. $f(x) = \dfrac{A}{B + Ce^x}$

32. $f(x) = \dfrac{1 - xe^x}{x + e^x}$

33. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$

34. $f(x) = \dfrac{ax + b}{cx + d}$

35–36 Find an equation of the tangent line to the given curve at the specified point.

35. $y = \dfrac{2x}{x + 1}$, $(1, 1)$

36. $y = e^x \cos x$, $(0, 1)$

37–38 Find equations of the tangent line and normal line to the given curve at the specified point.

37. $y = 2xe^x$, $(0, 0)$

38. $y = \dfrac{\sqrt{x}}{x + 1}$, $(4, 0.4)$

39–40 Find $f'(x)$ and $f''(x)$.

39. $f(x) = x^4 e^x$

40. $f(x) = \dfrac{x}{x^2 - 1}$

41. If $H(\theta) = \theta \sin\theta$, find $H'(\theta)$ and $H''(\theta)$.

42. If $f(t) = \csc t$, find $f''(\pi/6)$.

43. If $f(x) = xe^x$, find the nth derivative, $f^{(n)}(x)$.

44. If $g(x) = x/e^x$, find $g^{(n)}(x)$.

45. Prove that $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$.

46. Prove that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

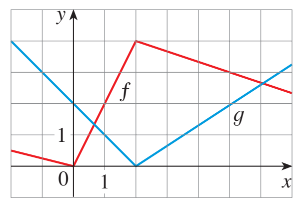
47. Prove that $\dfrac{d}{dx}(\cot x) = -\csc^2 x$.

48. Suppose that $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x) \sin x$ and $h(x) = (\cos x)/f(x)$. Find
(a) $g'(\pi/3)$ (b) $h'(\pi/3)$

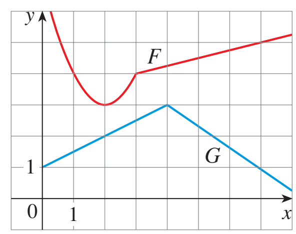
49. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.
(a) $(fg)'(5)$ (b) $(f/g)'(5)$ (c) $(g/f)'(5)$

50. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$.
(a) $h(x) = 5f(x) - 4g(x)$ (b) $h(x) = f(x)g(x)$
(c) $h(x) = \dfrac{f(x)}{g(x)}$ (d) $h(x) = \dfrac{g(x)}{1 + f(x)}$

51. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.
(a) Find $u'(1)$. (b) Find $v'(5)$.



52. Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.
(a) Find $P'(2)$. (b) Find $Q'(7)$.



53. If g is a differentiable function, find an expression for the derivative of each of the following functions.
(a) $y = xg(x)$ (b) $y = \dfrac{x}{g(x)}$ (c) $y = \dfrac{g(x)}{x}$

54. Insecticide resistance If the frequency of a gene for insecticide resistance is p, then its frequency in the next generation is given by the expression

$$\frac{p(1 + s)}{1 + sp}$$

where s is the reproductive advantage that this gene has over the wild type in the presence of the insecticide. Determine the rate at which the gene frequency in the next generation changes as s changes.

55. The **Michaelis-Menten equation** for the enzyme chymotrypsin is

$$v = \frac{0.14[S]}{0.015 + [S]}$$

where v is the rate of an enzymatic reaction and [S] is the concentration of a substrate S. Calculate $dv/d[S]$ and interpret it.

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56. The **biomass** $B(t)$ of a fish population is the total mass of the members of the population at time t. It is the product of the number of individuals $N(t)$ in the population and the average mass $M(t)$ of a fish at time t. In the case of guppies, breeding occurs continually. Suppose that at time $t = 4$ weeks the population is 820 guppies and is growing at a rate of 50 guppies per week, while the average mass is 1.2 g and is increasing at a rate of 0.14 g/week. At what rate is the biomass increasing when $t = 4$?

57. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

58. Sensitivity of the eye to brightness If R denotes the reaction of the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.

(a) Find the sensitivity.

(b) Illustrate part (a) by graphing both R and S as functions of x. Comment on the values of R and S at low levels of brightness. Is this what you would expect?

59. How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

60. Find equations of the tangent lines to the curve

$$y = \frac{x - 1}{x + 1}$$

that are parallel to the line $x - 2y = 2$.

61. (a) Use the Product Rule twice to prove that if f, g, and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.

(b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) Use part (b) to differentiate $y = e^{3x}$.

62. (a) If $F(x) = f(x)\,g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.

(b) Find similar formulas for F''' and $F^{(4)}$.

(c) Guess a formula for $F^{(n)}$.

63. (a) If g is differentiable, the **Reciprocal Rule** says that

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{[g(x)]^2}$$

Use the Quotient Rule to prove the Reciprocal Rule.

(b) Use the Reciprocal Rule to differentiate the function $y = 1/(x^4 + x^2 + 1)$.

(c) Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

for all positive integers n.

3.5 The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

See Section 1.3 for a review of composite functions.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$. We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule*. It seems plausible if we interpret derivatives as rates of change. Regard du/dx as the rate of change of u with respect to x, dy/du as the rate of change of y with respect to u, and dy/dx as the rate of change of y with respect to x. If u changes

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