EXAMPLE 12 | Differentiate $y = 3 \sin \theta + 4 \cos \theta$.

SOLUTION

$$\frac{dy}{d\theta} = 3 \frac{d}{d\theta} (\sin \theta) + 4 \frac{d}{d\theta} (\cos \theta) = 3 \cos \theta - 4 \sin \theta$$

EXAMPLE 13 | Find the 27th derivative of $\cos x$.

SOLUTION The first few derivatives of $f(x) = \cos x$ are as follows:

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

Looking for a pattern, we see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$

EXERCISES 3.3

1. (a) How is the number e defined?
(b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \qquad \text{and} \qquad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What fact allows you to do this?
(b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
(c) Which of the two functions in part (b) grows more rapidly when x is large?

3–32 Differentiate the function.

3. $f(x) = 186.5$

4. $f(x) = \sqrt{30}$

5. $f(x) = 5x - 1$

6. $F(x) = \frac{3}{4}x^8$

7. $f(x) = x^3 - 4x + 6$

8. $f(t) = \frac{1}{2}t^6 - 3t^4 + t$

9. $f(x) = x - 3 \sin x$

10. $y = \sin t + \pi \cos t$

11. $f(t) = \frac{1}{4}(t^4 + 8)$

12. $h(x) = (x - 2)(2x + 3)$

13. $A(s) = -\dfrac{12}{s^5}$

14. $B(y) = cy^{-6}$

15. $g(t) = 2t^{-3/4}$

16. $h(t) = \sqrt[4]{t} - 4e^t$

17. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}}$

18. $y = \sqrt{x}\,(x - 1)$

19. $F(x) = (\frac{1}{2}x)^5$

20. $f(x) = \dfrac{x^2 - 3x + 1}{x^2}$

21. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

22. $g(u) = \sqrt{2}\,u + \sqrt{3u}$

23. $y = 4\pi^2$

24. $L(\theta) = \dfrac{\sin \theta}{2} + \dfrac{c}{\theta}$

25. $g(y) = \dfrac{A}{y^{10}} + B \cos y$

26. $h(N) = rN\left(1 - \dfrac{N}{K}\right)$

27. $f(x) = k(a - x)(b - x)$

28. $F(v) = ae^v + \dfrac{b}{v} + \dfrac{c}{v^2}$

29. $u = \sqrt[5]{t} + 4\sqrt{t^5}$

30. $v = \left(\sqrt{x} + \dfrac{1}{\sqrt[3]{x}}\right)^2$

31. $G(y) = \dfrac{A}{y^{10}} + Be^y$

32. $y = e^{x+1} + 1$

33–34 Find an equation of the tangent line to the curve at the given point.

33. $y = \sqrt[4]{x}, \quad (1, 1)$

34. $y = x^4 + 2x^2 - x, \quad (1, 2)$

35–38 Find equations of the tangent line and normal line to the curve at the given point.

35. $y = 6 \cos x, \quad (\pi/3, 3)$

36. $y = x^2 - x^4, \quad (1, 0)$

37. $y = x^4 + 2e^x, \quad (0, 2)$

38. $y = (1 + 2x)^2, \quad (1, 9)$

39–40 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

39. $y = x + \sqrt{x}, \quad (1, 2)$

40. $y = 3x^2 - x^3, \quad (1, 2)$

41–42 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

41. $f(x) = 3x^{15} - 5x^3 + 3$

42. $f(x) = x + \dfrac{1}{x}$

43–46 Find the first and second derivatives of the function.

43. $f(x) = x^4 - 3x^3 + 16x$

44. $G(r) = \sqrt{r} + \sqrt[3]{r}$

45. $g(t) = 2 \cos t - 3 \sin t$

46. $h(t) = \sqrt{t} + 5 \sin t$

47. Fish growth Biologists have proposed a cubic polynomial to model the length L of rock bass at age A:

$$L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21$$

where L is measured in inches and A in years. (See Exercise 1.2.27.) Calculate

$$\left.\frac{dL}{dA}\right|_{A=12}$$

and interpret your answer.

48. Rain forest biodiversity The number of tree species S in a given area A in the Pasoh Forest Reserve in Malaysia has been modeled by the power function

$$S(A) = 0.882A^{0.842}$$

where A is measured in square meters. Find $S'(100)$ and interpret your answer.

(*Source:* Adapted from K. Kochummen et al., "Floristic Composition of Pasoh Forest Reserve, a Lowland Rain Forest in Peninsular Malaysia," *Journal of Tropical Forest Science* 3 (1991): 1–13.

49. Blood flow Refer to the law of laminar flow given in Example 9. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm^2, and viscosity $\eta = 0.027$.

(a) Find the velocity of the blood along the centerline $r = 0$, at radius $r = 0.005$ cm, and at the wall $r = R = 0.01$ cm.

(b) Find the velocity gradient at $r = 0$, $r = 0.005$, and $r = 0.01$.

(c) Where is the velocity the greatest? Where is the velocity changing most?

50. Invasive species often display a wave of advance as they colonize new areas. Mathematical models based on random dispersal and reproduction have demonstrated that the speed with which such waves move is given by the expression $2\sqrt{Dr}$, where r is the reproductive rate of individuals and D is a parameter quantifying dispersal. Calculate the derivative of the wave speed with respect to the reproductive rate r and explain its meaning.

51. The position function of a particle is given by $s = t^3 - 4.5t^2 - 7t$, $t \geqslant 0$.

(a) Find the velocity and acceleration of the particle.

(b) When does the particle reach a velocity of 5 m/s?

(c) When is the acceleration 0? What is the significance of this value of t?

52. If a ball is given a push so that it has an initial velocity of 5 m/s rolling down a certain inclined plane, then the distance it has rolled after t seconds is $s = 5t + 3t^2$.

(a) Find the velocity after 2 s.

(b) How long does it take for the velocity to reach 35 m/s?

53. (a) A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm and it wants to know how the area $A(x)$ of a wafer changes when the side length x changes. Find $A'(15)$ and explain its meaning in this situation.

(b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx. How can you approximate the resulting change in area ΔA if Δx is small?

54. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If V is the volume of such a cube with side length x, calculate dV/dx when $x = 3$ mm and explain its meaning.

(b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 53(b).

55. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from

(i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1

(b) Find the instantaneous rate of change when $r = 2$.

(c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

56. (a) **Cell growth** The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\ \mu\text{m} = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from
(i) 5 to 8 μm (ii) 5 to 6 μm (iii) 5 to 5.1 μm
(b) Find the instantaneous rate of change of V with respect to r when $r = 5\ \mu$m.
(c) Show that the rate of change of the volume of a cell with respect to its radius is equal to its surface area ($S = 4\pi r^2$). Explain geometrically why this result is true. Argue by analogy with Exercise 55(c).

57. Find $\dfrac{d^{99}}{dx^{99}}(\sin x)$.

58. Find the nth derivative of each function by calculating the first few derivatives and observing the pattern that occurs.
(a) $f(x) = x^n$ (b) $f(x) = 1/x$

59. For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

60. For what values of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

61. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

62. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

63. Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and parallel to the line $12x - y = 1$.

64. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

65. Find an equation of the normal line to the parabola $y = x^2 - 5x + 4$ that is parallel to the line $x - 3y = 5$.

66. Where does the normal line to the parabola $y = x - x^2$ at the point $(1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

67. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines TOUCH ~~intersect~~ the parabola.

68. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
(b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

69. Use the definition of a derivative to show that if $f(x) = 1/x$, then $f'(x) = -1/x^2$. (This proves the Power Rule for the case $n = -1$.)

70. Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

71. Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1, 1)$ has equation $y = 3x - 2$.

72. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.

73. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

74. A tangent line is drawn to the hyperbola $xy = c$ at a point P.
(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P.
(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

75. Evaluate $\displaystyle\lim_{x\to 1} \frac{x^{1000} - 1}{x - 1}$.

76. Draw a diagram showing two perpendicular lines that intersect on the y-axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

3.4 The Product and Quotient Rules

The formulas of this section enable us to differentiate new functions formed from old functions by multiplication or division.

The Product Rule

By analogy with the Sum and Difference Rules, one might be tempted to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$. But