EXERCISES 2.4

1. Given that

$$\lim_{x \to 2} f(x) = 4 \qquad \lim_{x \to 2} g(x) = -2 \qquad \lim_{x \to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 2} [g(x)]^3$

(c) $\lim_{x \to 2} \sqrt{f(x)}$ (d) $\lim_{x \to 2} \dfrac{3f(x)}{g(x)}$

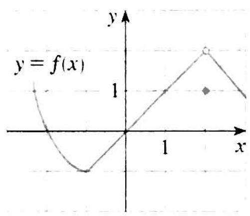
(e) $\lim_{x \to 2} \dfrac{g(x)}{h(x)}$ (f) $\lim_{x \to 2} \dfrac{g(x)h(x)}{f(x)}$

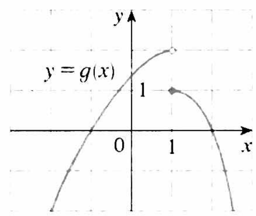
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + g(x)]$ (b) $\lim_{x \to 1} [f(x) + g(x)]$

(c) $\lim_{x \to 0} [f(x)g(x)]$ (d) $\lim_{x \to -1} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} [x^3 f(x)]$ (f) $\lim_{x \to 1} \sqrt{3 + f(x)}$





3–7 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to -2} (3x^4 + 2x^2 - x + 1)$ **4.** $\lim_{t \to -1} (t^2 + 1)^3(t + 3)^5$

5. $\lim_{x \to 2} \sqrt{\dfrac{2x^2 + 1}{3x - 2}}$ **6.** $\lim_{x \to 0} \dfrac{\cos^4 x}{5 + 2x^3}$

7. $\lim_{\theta \to \pi/2} \theta \sin \theta$

8. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

9–24 Evaluate the limit, if it exists.

9. $\lim_{x \to 5} \dfrac{x^2 - 6x + 5}{x - 5}$ **10.** $\lim_{x \to 4} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

11. $\lim_{x \to 5} \dfrac{x^2 - 5x + 6}{x - 5}$ **12.** $\lim_{x \to -1} \dfrac{2x^2 + 3x + 1}{x^2 - 2x - 3}$

13. $\lim_{t \to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3}$ **14.** $\lim_{x \to -1} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

15. $\lim_{h \to 0} \dfrac{(4 + h)^2 - 16}{h}$ **16.** $\lim_{h \to 0} \dfrac{(2 + h)^3 - 8}{h}$

17. $\lim_{x \to -2} \dfrac{x + 2}{x^3 + 8}$ **18.** $\lim_{h \to 0} \dfrac{\sqrt{1 + h} - 1}{h}$

19. $\lim_{x \to -4} \dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x}$ **20.** $\lim_{x \to -1} \dfrac{x^2 + 2x + 1}{x^4 - 1}$

21. $\lim_{x \to 16} \dfrac{4 - \sqrt{x}}{16x - x^2}$ **22.** $\lim_{t \to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2 + t} \right)$

23. $\lim_{t \to 0} \left(\dfrac{1}{t\sqrt{1 + t}} - \dfrac{1}{t} \right)$ **24.** $\lim_{x \to -4} \dfrac{\sqrt{x^2 + 9} - 5}{x + 4}$

25. (a) Estimate the value of

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

26. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

27. Use the Squeeze Theorem to show that $\lim_{x \to 0} (x^2 \cos 20\pi x) = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

28. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

29. If $4x - 9 \leqslant f(x) \leqslant x^2 - 4x + 7$ for $x \geqslant 0$, find $\lim_{x \to 4} f(x)$.

30. If $2x \leqslant g(x) \leqslant x^4 - x^2 + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$.

31. Prove that $\lim_{x \to 0} x^4 \cos \dfrac{2}{x} = 0$.

32. Gene regulation Genes produce molecules called mRNA that go on to produce proteins. High concentrations of protein inhibit the production of mRNA, leading to stable gene regulation. This process has been modeled (see Section 10.3) to show that the concentration of mRNA over time is given by the equation

$$m(t) = \tfrac{1}{2}e^{-t}(\sin t - \cos t) + \tfrac{1}{2}$$

(a) Evaluate $\lim_{t \to 0} m(t)$ and interpret your result.
(b) Use the Squeeze Theorem to evaluate $\lim_{t \to \infty} m(t)$ and interpret your result.

33–36 Find the limit, if it exists. If the limit does not exist, explain why.

33. $\lim_{x \to 3} (2x + |x - 3|)$

34. $\lim_{x \to -6} \dfrac{2x + 12}{|x + 6|}$

35. $\lim_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

36. $\lim_{x \to -2} \dfrac{2 - |x|}{2 + x}$

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37. Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.

(a) Find
(i) $\lim_{x \to 2^+} g(x)$ (ii) $\lim_{x \to 2^-} g(x)$
(b) Does $\lim_{x \to 2} g(x)$ exist?
(c) Sketch the graph of g.

38. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
(b) Does $\lim_{x \to 1} f(x)$ exist?
(c) Sketch the graph of f.

39–44 Find the limit.

39. $\lim_{x \to 0} \dfrac{\sin 3x}{x}$

40. $\lim_{x \to 0} \dfrac{\sin 4x}{\sin 6x}$

41. $\lim_{t \to 0} \dfrac{\tan 6t}{\sin 2t}$

42. $\lim_{t \to 0} \dfrac{\sin^2 3t}{t^2}$

43. $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta + \tan \theta}$

44. $\lim_{x \to 0} x \cot x$

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45. (a) If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.
(b) If r is a rational function, use part (a) to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

46. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

47. To prove that sine has the Direct Substitution Property we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. If we let $h = x - a$, then $x = a + h$ and $x \to a \iff h \to 0$. So an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (5) to show that this is true.

48. Prove that cosine has the Direct Substitution Property.

49. If $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x \to 1} f(x)$.

50. If $\lim_{x \to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.

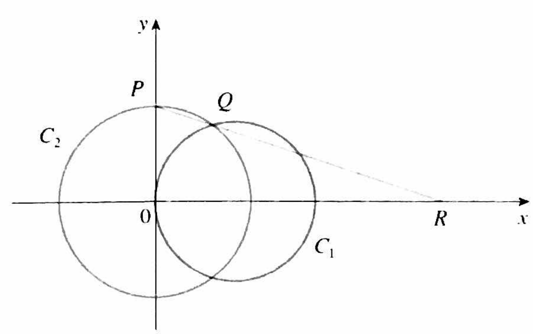
(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 0} \dfrac{f(x)}{x}$

51. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

52. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



10. Sketch the graph of an example of a function f that satisfies all of the following conditions:

$\lim_{x\to-\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to-3} f(x) = \infty$,

$\lim_{x\to3^-} f(x) = -\infty$, $\lim_{x\to3^+} f(x) = 2$,

f is continuous from the right at 3

11–28 Find the limit.

11. $\lim_{x\to\infty} \dfrac{1-x}{2+5x}$

12. $\lim_{t\to\infty} 3^{-2t}$

13. $\lim_{x\to1} e^{x^3-x}$

14. $\lim_{x\to3} \dfrac{x^2-9}{x^2+2x-3}$

15. $\lim_{x\to-3} \dfrac{x^2-9}{x^2+2x-3}$

16. $\lim_{x\to1^+} \dfrac{x^2-9}{x^2+2x-3}$

17. $\lim_{h\to0} \dfrac{(h-1)^3+1}{h}$

18. $\lim_{t\to2} \dfrac{t^2-4}{t^3-8}$

19. $\lim_{r\to9} \dfrac{\sqrt{r}}{(r-9)^4}$

20. $\lim_{v\to4^+} \dfrac{4-v}{|4-v|}$

21. $\lim_{u\to1} \dfrac{u^4-1}{u^3+5u^2-6u}$

22. $\lim_{x\to3} \dfrac{\sqrt{x+6}-x}{x^3-3x^2}$

23. $\lim_{x\to\pi^-} \ln(\sin x)$

24. $\lim_{x\to-\infty} \dfrac{1-2x^2-x^4}{5+x-3x^4}$

25. $\lim_{x\to\infty} \dfrac{\sqrt{x^2-9}}{2x-6}$

26. $\lim_{x\to\infty} e^{x-x^2}$

27. $\lim_{x\to\infty} \left(\sqrt{x^2+4x+1}-x\right)$

28. $\lim_{x\to1} \left(\dfrac{1}{x-1}+\dfrac{1}{x^2-3x+2}\right)$

29. The **Michaelis-Menten equation** for the rate v of the enzymatic reaction of the concentration [S] of a substrate S, in the case of the enzyme pepsin, is

$$v = \frac{0.50[S]}{3.0\times10^{-4}+[S]}$$

What is $\lim_{[S]\to\infty} v$? What is the meaning of the limit in this context?

30. Prove that $\lim_{x\to0} x^2\cos(1/x^2) = 0$.

31. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x<0 \\ 3-x & \text{if } 0\le x<3 \\ (x-3)^2 & \text{if } x>3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x\to0^+} f(x)$ (ii) $\lim_{x\to0^-} f(x)$ (iii) $\lim_{x\to0} f(x)$

(iv) $\lim_{x\to3^-} f(x)$ (v) $\lim_{x\to3^+} f(x)$ (vi) $\lim_{x\to3} f(x)$

(b) Where is f discontinuous?

(c) Sketch the graph of f.

32. Show that each function is continuous on its domain. State the domain.

(a) $g(x) = \dfrac{\sqrt{x^2-9}}{x^2-2}$ (b) $h(x) = xe^{\sin x}$

33–34 Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

33. $2x^3+x^2+2=0$, $(-2,-1)$

34. $e^{-x^2}=x$, $(0,1)$

CASE STUDY 2a Hosts, Parasites, and Time-Travel

We are studying a model for the interaction between *Daphnia* and its parasite. Recall that there are two possible host genotypes (A and a) and two possible parasite genotypes (B and b). Parasites of type B can infect only hosts of type A, while parasites of type b can infect only hosts of type a. Here we will take equations that will be obtained in Case Studies 2b and 2d to explore the biological predictions that can be obtained from them.

In Case Study 2d we will derive the functions

(1a) $$q(t) = \tfrac{1}{2} + M_q\cos(ct-\phi_q)$$

(1b) $$p(t) = \tfrac{1}{2} + M_p\cos(ct-\phi_p)$$