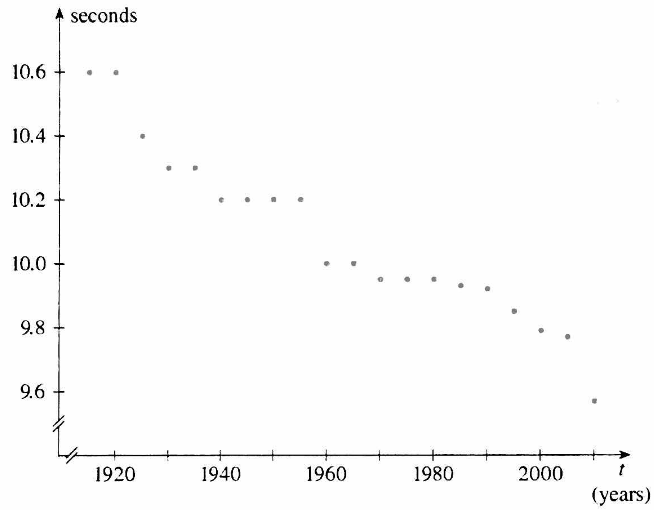
EXERCISES 2.1

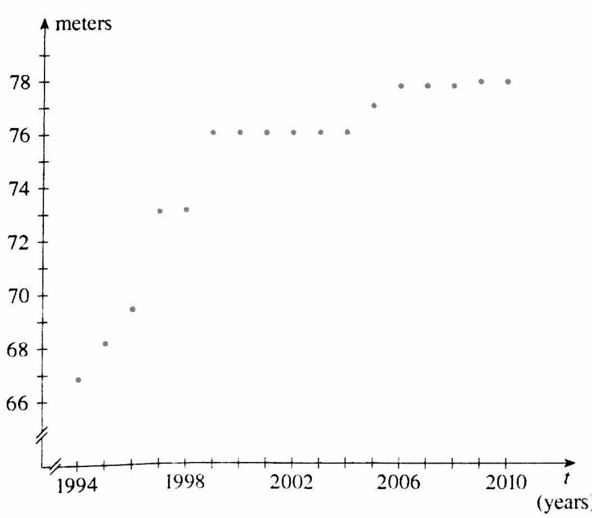
1. (a) What is a sequence?
(b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
(c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

2. (a) What is a convergent sequence? Give two examples.
(b) What is a divergent sequence? Give two examples.

3. World record sprint times The graph plots the sequence of the world record times for the men's 100-meter sprint every five years t. Do you think that this sequence has a nonzero limit as $t \to \infty$? What would that mean for this sporting event?



4. World record hammer throws The graph plots the sequence of the world record distances for the women's hammer throw by year t.
(a) Explain what it would mean for this sporting event if the sequence does not have a limit as $t \to \infty$.
(b) Do you think this sequence is convergent or divergent? Explain.



5–8 Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

5. $a_n = \dfrac{n^2}{2n + 3n^2}$

6. $a_n = 4 - \dfrac{2}{n} + \dfrac{3}{n^2}$

7. $a_n = 3 + \left(-\frac{2}{3}\right)^n$

8. $a_n = \dfrac{n}{\sqrt{n} + 1}$

9–26 Determine whether the sequence is convergent or divergent. If it is convergent, find the limit.

9. $a_n = \dfrac{1}{3n^4}$

10. $a_n = \dfrac{5}{3^n}$

11. $a_n = \dfrac{2n^2 + n - 1}{n^2}$

12. $a_n = \dfrac{n^3 - 1}{n}$

13. $a_n = \dfrac{3 + 5n}{2 + 7n}$

14. $a_n = \dfrac{n^3 - 1}{n^3 + 1}$

15. $a_n = 1 - (0.2)^n$

16. $a_n = 2^{-n} + 6^{-n}$

17. $a_n = \dfrac{n^2}{\sqrt{n^3 + 4n}}$

18. $a_n = \sin(n\pi/2)$

19. $a_n = \cos(n\pi/2)$

20. $a_n = \dfrac{\pi^n}{3^n}$

21. $a_n = \dfrac{10^n}{1 + 9^n}$

22. $a_n = \dfrac{\sqrt[3]{n}}{\sqrt{n} + \sqrt[4]{n}}$

23. $a_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$

24. $a_n = \dfrac{3^{n+2}}{5^n}$

25. $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

26. $a_n = \ln(n + 1) - \ln n$

27–34 Calculate, to four decimal places, the first eight terms of the recursive sequence. Does it appear to be convergent? If so, guess the value of the limit. Then assume the limit exists and determine its exact value.

27. $a_1 = 1,\ a_{n+1} = \frac{1}{2}a_n + 1$

28. $a_1 = 2,\ a_{n+1} = 1 - \frac{1}{3}a_n$

29. $a_1 = 2,\ a_{n+1} = 2a_n - 1$

30. $a_1 = 1,\ a_{n+1} = \sqrt{5a_n}$

31. $a_1 = 1,\ a_{n+1} = \dfrac{6}{1 + a_n}$

32. $a_1 = 3,\ a_{n+1} = 8 - a_n$

33. $a_1 = 1,\ a_{n+1} = \sqrt{2 + a_n}$

34. $a_1 = 100,\ a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{25}{a_n}\right)$

35. Antibiotic pharmacokinetics A doctor prescribes a 100-mg antibiotic tablet to be taken every eight hours. Just before each tablet is taken, 20% of the drug present in the preceding time step remains in the body.
(a) How much of the drug is in the body just after the second tablet is taken? After the third tablet?
(b) If Q_n is the quantity of the antibiotic in the body just after the nth tablet is taken, write a difference equation that expresses Q_{n+1} in terms of Q_n.
(c) Find a formula for Q_n as a function of n.
(d) What quantity of the antibiotic remains in the body in the long run?

36. Drug pharmacokinetics A patient is injected with a drug every 12 hours. Immediately before each injection the concentration of the drug has been reduced by 90% and the new dose increases the concentration by 1.5 mg/mL.
(a) What is the concentration after three doses?
(b) If C_n is the concentration after the nth dose, write a difference equation that expresses C_{n+1} in terms of C_n.
(c) Find a formula for C_n as a function of n.
(d) What is the limiting value of the concentration?

37. Drug pharmacokinetics A patient takes 150 mg of a drug at the same time every day. Just before each tablet is taken, 5% of the drug present in the preceding time step remains in the body.
(a) What quantity of the drug is in the body after the third tablet? After the nth tablet?
(b) What quantity of the drug remains in the body in the long run?

38. Insulin injection After injection of a dose D of insulin, the concentration of insulin in a patient's system decays exponentially and so it can be written as De^{-at}, where t represents time in hours and a is a positive constant.
(a) If a dose D is injected every T hours, write an expression for the sum of the residual concentrations just before the $(n + 1)$st injection.
(b) Determine the limiting pre-injection concentration.
(c) If the concentration of insulin must always remain at or above a critical value C, determine a minimal dosage D in terms of C, a, and T.

39. Let $x = 0.99999\ldots$.
(a) Do you think that $x < 1$ or $x = 1$?
(b) Sum a geometric series to find the value of x.
(c) How many decimal representations does the number 1 have?
(d) Which numbers have more than one decimal representation?

40. A sequence is defined recursively by $a_n = (5 - n)a_{n-1}$, $a_1 = 1$. Find the sum of all the terms of the sequence.

41–46 Express the number as a ratio of integers.

41. $0.\overline{8} = 0.8888\ldots$ **42.** $0.\overline{46} = 0.46464646\ldots$

43. $2.\overline{516} = 2.516516516\ldots$

44. $10.1\overline{35} = 10.135353535\ldots$

45. $1.53\overline{42}$ **46.** $7.\overline{12345}$

47–52 Logistic equation Plot enough terms of the logistic difference equation $x_{t+1} = cx_t(1 - x_t)$ to see how the terms behave. Does the sequence appear to be convergent? If so, estimate the limit and then, assuming the limit exists, calculate its exact value. If not, describe the behavior of the terms.

47. $x_0 = 0.1,\ c = 2$ **48.** $x_0 = 0.8,\ c = 2.6$

49. $x_0 = 0.2,\ c = 3.2$ **50.** $x_0 = 0.4,\ c = 3.5$

51. $x_0 = 0.1,\ c = 3.8$ **52.** $x_0 = 0.6,\ c = 3.9$

53. Logistic equation: Dependence on initial values Compare plots of the first 20 terms of the logistic equation $x_{t+1} = \frac{1}{4}x_t(1 - x_t)$ for the initial values $x_0 = 0.2$ and $x_0 = 0.2001$. When the initial value changes slightly, how does the solution change?

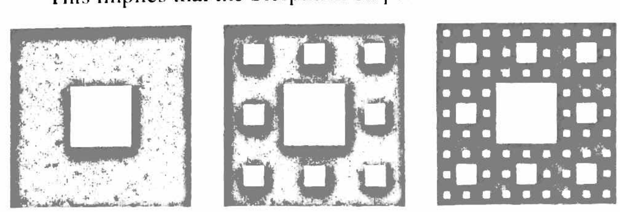
54. Logistic equation: Dependence on initial values Repeat Exercise 53 for the equation $x_{t+1} = 4x_t(1 - x_t)$ and compare with the results of Exercise 53. [This behavior is another part of what it means to be *chaotic*.]

55–58 The **Ricker equation** $x_{t+1} = cx_te^{-x_t}$ was introduced in Exercise 1.6.32. Plot enough terms of the Ricker equation to see how the terms behave. Does the sequence appear to be convergent? If so, estimate the limit and then, assuming the limit exists, calculate its exact value. If not, describe the behavior of the terms.

55. $x_0 = 0.2,\ c = 2$ **56.** $x_0 = 0.4,\ c = 3$

57. $x_0 = 0.8,\ c = 10$ **58.** $x_0 = 0.9,\ c = 20$

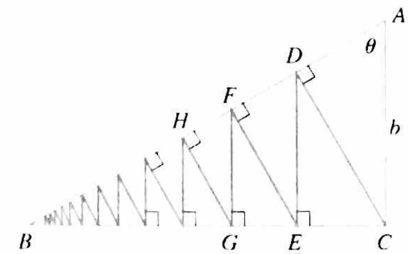
59. The **Sierpinski carpet** is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.



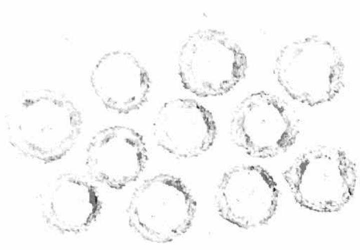
60. A right triangle ABC is given with $\angle A = \theta$ and $|AC| = b$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, $EF \perp AB$, and this process is continued indefinitely, as shown in the figure. Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

in terms of b and θ.



■ PROJECT Modeling the Dynamics of Viral Infections[1]



Chris Bjornberg / Science Source

A patient is infected with a virus that triples its numbers every hour. The immune system eventually kicks in and reduces the replication rate by a factor of $\frac{1}{2}$ in addition to killing 500,000 virus particles per hour, but this doesn't happen until the viral load reaches two million copies.

To combat the infection, the infected person receives hourly doses of an antiviral drug. This drug further reduces the replication rate, to a value of 1.25, and the immune system and the drug together can kill 25,000,000 copies of the virus per hour.

Let's model the phases of the infection using a discrete-time difference equation.

1. What is the recursion for the number of viral particles in the absence of treatment and before the immune response starts? What is the equation for the number of viral particles as a function of time?

2. How long does it take for the immune system to be activated after the infection starts? Derive an equation for this length of time for an arbitrary initial number of viral particles.

3. What is the recursion for the number of viral particles after the immune response has begun, but before the drug is used?

4. What is the condition for the viral population size to decrease over time solely because of the immune system? Is this possible?

5. What is the recursion for the number of viral particles in the presence of both the immune response and the drug?

6. What is the condition for the viral population size to decrease over time when the drug and immune system are both acting? Is this possible?

7. If an individual is infected by one virus, how much time do you have to start drug treatment in order to control the infection?

8. Often the outcome of an infection depends on the number of viral particles causing the infection. Derive an expression for the amount of time it takes from the initial infection for an individual to reach the critical viral load—beyond which control of the infection is impossible—as a function of the initial number of particles n_0 and an arbitrary initial replication rate R.

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1. Adapted from F. Giordano et al., *A First Course in Mathematical Modeling* (Belmont, CA: Cengage Learning, 2014).