4.

The indirect utility function for a consumer with a utility function \( U(x, y) \) is defined to be a function \( V(p_1, p_2, m) \) such that \( V(p_1, p_2, m) \) is the maximum utility \( U(x, y) \) that could be attained subject to the constraint that the consumer can afford \((x_1, x_2)\) at the price \((p_1, p_2)\) with income \(m\).

a. Find the indirect utility function for someone with the utility function \( U(x, y) = 2x + y \)

In the utility function above, \( x \) and \( y \) are substitutes.

The Consumer Maximization Problem is as follows:

\[
\max_{\{x, y\}} \{2x + y\}
\]

subject to

\[ p_1x + p_2y = m \]

Where \( p_1 \) and \( p_2 \) are the price \( x \) and \( y \) of respectively and \( m \) is income.

The Marginal Rates of Substitution (MRS):

\[
MRS = -\left( \frac{MU_x}{MU_y} \right) = - \left[ \frac{\partial U(x, y)}{\partial x} \right] \left[ \frac{\partial U(x, y)}{\partial y} \right]
\]

Where \( MU_x \) is the marginal utility of \( x \) and \( MU_y \) is the marginal utility of \( y \)

\[
\frac{\partial U(x, y)}{\partial x} = 2
\]

\[
\frac{\partial U(x, y)}{\partial y} = 1
\]

Hence,

\[
MRS = - \frac{2}{1} = -2
\]

The budget constraint:

\[ p_1x + p_2y = m \]
Rewriting the budget constraint:

\[ y = \frac{m}{p_2} - \left( \frac{p_1}{p_2} \right) x \]

The slope of the budget constraint is \(- \left( \frac{p_1}{p_2} \right)\).

Since the utility function is linear in terms of the commodity \(x\) and \(y\) we need to compare the absolute value of the marginal rates of substitution with the absolute value of the slope of the budget constraint. The **Consumer Maximization Problem** above has corner solutions since the utility function is linear in terms of \(x\) and \(y\).

In particular, we need to compare \(|MRS|\) with \(\left| \frac{p_1}{p_2} \right|\) where \(|.|\) represents the absolute value.

\[ |MRS| = |-2| = 2 \]

\[ |\text{Slope of the Budget Constraint}| = \left| - \left( \frac{p_1}{p_2} \right) \right| = \frac{p_1}{p_2} \]

Since we don’t know the value of \(p_1\) and \(p_2\) then we need to study the following cases.

**Case 1: \( \frac{p_1}{p_2} > 2 \)**

In this case, the slope of the budget constraint is **steeper** than the marginal rates of substitution MRS.

Hence, the **optimal allocation** \((x^*, y^*)\):

\[ x^* = 0, \quad y^* = \frac{m}{p_2} \]

Note that when \(x^* = 0\) we can use the budget constraint to pin down \(y^*\)

In particular,

\[ p_2y^* = m \Rightarrow y^* = \frac{m}{p_2} \]

As a result,

\[ (x^*, y^*) = \left( 0, \frac{m}{p_2} \right) \]

The indirect utility function:

\[ V(p_1, p_2, m) = U(x^*, y^*) = y^* = \frac{m}{p_2} \]

Note that \((x^*, y^*)\) is the **optimal allocation** not any consumption bundle.

Hence,

\[ V(p_1, p_2, m) = \frac{m}{p_2} \]

**Case 2: \( \frac{p_1}{p_2} < 2 \)**

In this case, the slope of the budget constraint is **flatter** than the marginal rates of substitution MRS.

Hence, the **optimal allocation** \((x^*, y^*)\):

\[ x^* = \frac{m}{p_1}, \quad y^* = 0 \]

Note that when \(y^* = 0\) we can use the budget constraint to pin down \(x^*\)
In particular,

\[ p_1 x^* = m \Rightarrow x^* = \frac{m}{p_1} \]

As a result,

\[ (x^*, y^*) = \left( \frac{m}{p_1}, 0 \right) \]

The indirect utility function:

\[ V(p_1, p_2, m) = U(x^*, y^*) = 2x^* = \frac{2m}{p_1} \]

Note that \((x^*, y^*)\) is the **optimal allocation** not any consumption bundle. Hence,

\[ V(p_1, p_2, m) = \frac{2m}{p_1} \]

**Case 3: \( \frac{p_1}{p_2} = 2 \)**

In the case, \(x^*\) and \(y^*\) can be any consumption bundle that satisfies the budget constraint. Hence, any \((x^*, y^*)\) that satisfies the following:

\[ p_1 x^* + p_2 y^* = m \]

Hence, we could solve the constraint above for \(y^*\) as follows:

\[ y^* = \frac{m}{p_2} - \left( \frac{p_1}{p_2} \right) x^* \]

The indirect utility function:

\[ V(p_1, p_2, m) = U(x^*, y^*) = 2x^* + y^* \]

Plug in \(y^*\) above, we have

\[ V(p_1, p_2, m) = 2x^* + \left( \frac{m}{p_2} - \left( \frac{p_1}{p_2} \right) x^* \right) \]

Rearranging terms, we have the following:

\[ V(p_1, p_2, m) = 2x^* - \left( \frac{p_1}{p_2} \right) x^* + \frac{m}{p_2} \]

\[ V(p_1, p_2, m) = \left[ 2 - \frac{p_1}{p_2} \right] x^* + \frac{m}{p_2} \]

\[ V(p_1, p_2, m) = \frac{m}{p_2} + \left[ \frac{2p_2 - p_1}{p_2} \right] x^* \text{ for any } x^* \geq 0 \]
b. Find the indirect utility function for someone with the utility function \( U(x, y) = \min(2x, y) \)

In the utility function above, the commodity \( x \) and \( y \) are compliments.

The **Consumer Maximization Problem** is as follows:

\[
\max_{x, y} \{ U(x, y) \}
\]

where \( U(x, y) = \min(2x, y) \)

subject to

\[ p_1 x + p_2 y = m \]

Where \( p_1 \) and \( p_2 \) are the price \( x \) and \( y \) of respectively and \( m \) is income.

The optimal allocation \((x^*, y^*)\) must satisfies \( 2x^* = y^* \) since it is the cheapest way to achieve

Then, we can plug \( y^* \) into the budget constraint

Hence,

\[ p_1 x^* + p_2 (2x^*) = m \]

Factorize out \( x^* \), we then have as follows:

\[ (p_1 + 2p_2) x^* = m \]

\[ x^* = \frac{m}{p_1 + 2p_2} \]

And,

\[ y^* = 2x^* = 2 \left( \frac{m}{p_1 + 2p_2} \right) \]

As a result, the optimal allocation is as follows:

\[ (x^*, y^*) = \left( \frac{m}{p_1 + 2p_2}, \frac{2m}{p_1 + 2p_2} \right) \]

The indirect utility function:

\[ V(p_1, p_2, m) = U(x^*, y^*) = \min \left( 2 \left[ \frac{m}{p_1 + 2p_2} \right], \frac{2m}{p_1 + 2p_2} \right) \]

\[ V(p_1, p_2, m) = U(x^*, y^*) = \min \left( \frac{2m}{p_1 + 2p_2}, \frac{2m}{p_1 + 2p_2} \right) = \frac{2m}{p_1 + 2p_2} \]

Note that \((x^*, y^*)\) is the **optimal allocation** not any consumption bundle.

Hence, we have the following:

\[ V(p_1, p_2, m) = \frac{2m}{p_1 + 2p_2} \]