Math 22
Midterm 2 Review Problems

Note: these problems are not a complete representation of all topics that may appear on midterm 2.

1. Determine the domain and range of the following functions:
   a. $f(x,y) = \sqrt{3-x^2-y^2}$
   b. $g(x,y) = \sqrt{36-9x^2-4y^2}$
   c. $h(x,y) = 10+\ln(9-x^2-9y^2)$
   d. $F(x,y,z) = \left(16-x^2-y^2-z^2\right)^{1/4}$

2. Sketch some level curves and the graphs of the following functions:
   a. $f(x,y) = \sqrt{3-x^2-y^2}$
   b. $g(x,y) = 3-x^2-y^2$
   c. $h(x,y) = \cos x$

3. Find the limit, if it exists, or show that the limit does not exist:
   a. $\displaystyle\lim_{(x,y)\to(2,1)} \frac{xy}{x^2+3y^2}$
   b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+3y^2}$
   c. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$
   d. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

4. Find the first and second partial derivatives of the following functions:
   a. $f(x,y) = x^5y^4+2x^4y^5+x^3y^6-3xy^2+10$
   b. $w=\sin(\alpha)\cos(\beta)$
   c. $u=xz-5x^2y^3z^4$
   d. $g(x,y,z,t) = \dfrac{xy^2}{t+2z}$
   e. $u=x^{y/z}$

5. For the function in problem 4a above, show directly that $f_{xxy} = f_{xyx} = f_{yxx}$.

6. Verify that the function $u=(x^2+y^2+z^2)^{-1/2}$ is a solution to the three-dimensional Laplace equation $u_{xx}+u_{yy}+u_{zz} = 0$.

7. Let f be a differentiable function of one variable.

   a. Show that $z = f(xy)$ is a solution to the Partial Differential Equation $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$.

   b. Show that $z = f(x-y)$ is a solution to the PDE $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

   c. Show that $z = f(x^2+y^2)$ is a solution to the PDE $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$

8. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$:

   a. $\sin(xyz) = x+y+z$

   b. $yz = \ln(x+z)$

9. Find the equation of the tangent plane to the graph of the following functions at the given point.

   a. $f(x,y) = 3(x-1)^2 + 2(y+3)^2 + 7$ at $(2,-2,12)$

   b. $g(x,y) = \sqrt{36-9x^2-4y^2}$ at $(1,\sqrt{2}/2,5)$

   c. $h(x,y) = 10+\ln(9-x^2-9y^2)$ at $(2\sqrt{2},0,10)$

10. Find the Linearization of the following functions at the given point.

    a. $f(x,y) = 3(x-1)^2 + 2(y+3)^2 + 7$ at $(2,-2)$

    b. $g(x,y) = \sqrt{36-9x^2-4y^2}$ at $(1,\sqrt{2}/2)$

    c. $h(x,y) = 10+\ln(9-x^2-9y^2)$ at $(2\sqrt{2},\ 0)$

11. Use the Linearizations found in problem 11 to approximate the following functions at the given points.

    a. $f(x,y) = 3(x-1)^2 + 2(y+3)^2 + 7$, estimate $f(2.01,-1.99)$.

    b. $g(x,y) = \sqrt{36-9x^2-4y^2}$, estimate $g(1.001,.706)$. (Use $\sqrt{2}/2 = .707$, so that $\Delta y = -.001$)

    c. $h(x,y) = 10+\ln(9-x^2-9y^2)$, estimate $h(2.9,.1)$. (Use $2\sqrt{2} = 2.8$, so that $\Delta x = .1$)

12. Given that $\frac{1}{w} = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$, use implicit differentiation to find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$, then write the total differential dw. Use this expression for dw to estimate the error in the quantity w when the quantities x, y, and z are measured to have values $x = y = z = 1$, with errors not exceeding $.01$.

13. Use the chain rule to find the indicated partial derivative(s).

    a. $z = e^{x/y}$, $x = s/t$, $y = t/s$. Find $\partial z/\partial s$, and $\partial z/\partial t$.

    b. $u = x^2 + yz$, $x = pr\cos\theta$, $y = pr\sin\theta$, $z = p+r$. Find $\partial u/\partial p$, $\partial u/\partial r$, and $\partial u/\partial \theta$.

    c. $q = \sqrt{x^2+y^2+z^2}$, $x = t^2$, $y = \cos t$, $z = \sin t$. Find dq/dt.

14. Given $x+y+z = \ln(xyz)$, use implicit differentiation to find $\partial z/\partial x$, and $\partial z/\partial y$.

15. Find the directional derivative $f(x,\ y,\ z) = \tan(x+2y+3z)$ at $(-5,1,1)$ in the direction $\mathbf{v} = \langle 3, 2, \sqrt{3} \rangle$. Also find the directions of fastest increase and decrease of f at the same point, and find the rates of change in those directions.

16. Let $f(x,y) = ye^{-xy}$. Determine all unit vectors $\mathbf{u} = < a, b >$ such that $D_{\mathbf{u}} f(0,2) = 1$.

17. Let $F(x,y,z)$ be a differentiable function, and let C be a curve lying on the level surface $F(x,y,z) = k$. Suppose C is parameterized by the vector function $\mathbf{r}(t) = < f(t), g(t), h(t) >$. Use the chain rule to show that at each point (x,y,z) along the curve C, the gradient vector $\nabla F(x,y,z)$ is perpendicular to the tangent vector to C.

18. Find and classify all critical points of the function $f(x,y) = 2x^3 - 9x^2 + 12x + y^4 - 2y^2$.

19. Determine the absolute minimum and maximum of the function $f(x,y) = 2x^3 - 9x^2 + 12x + y^4 - 2y^2$ on the unit square in the *xy*-plane with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.

20. Find extrema of the following functions subject to the given constraint.
   a. $f(x,y) = 4x + 6y$, $x^2 + y^2 = 13$
   b. $g(x,y,z) = 8x - 4z$, $x^2 + 10y^2 + z^2 = 5$

21. Write the equation of the tangent plane to the surface $x^2 + y^2 - z^2 = 1$ at the point $(1,-1,1)$.