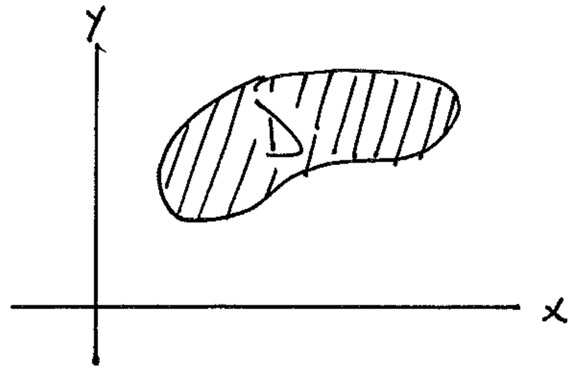
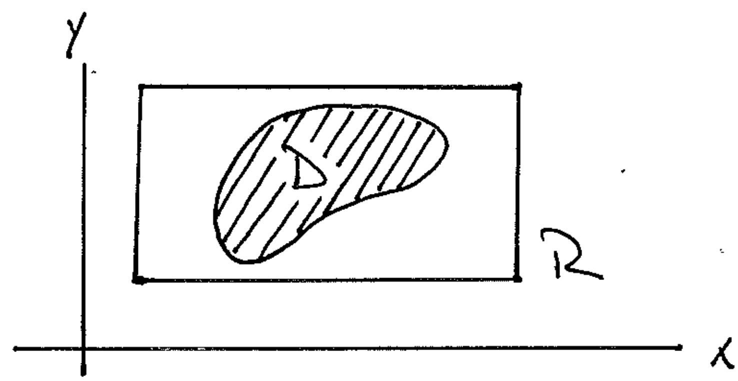
(15.3) More General Regions

We wish to define the integral of $f(x,y)$ over non-rectangular regions.



To do this we suppose R is a rectangular region containing D.

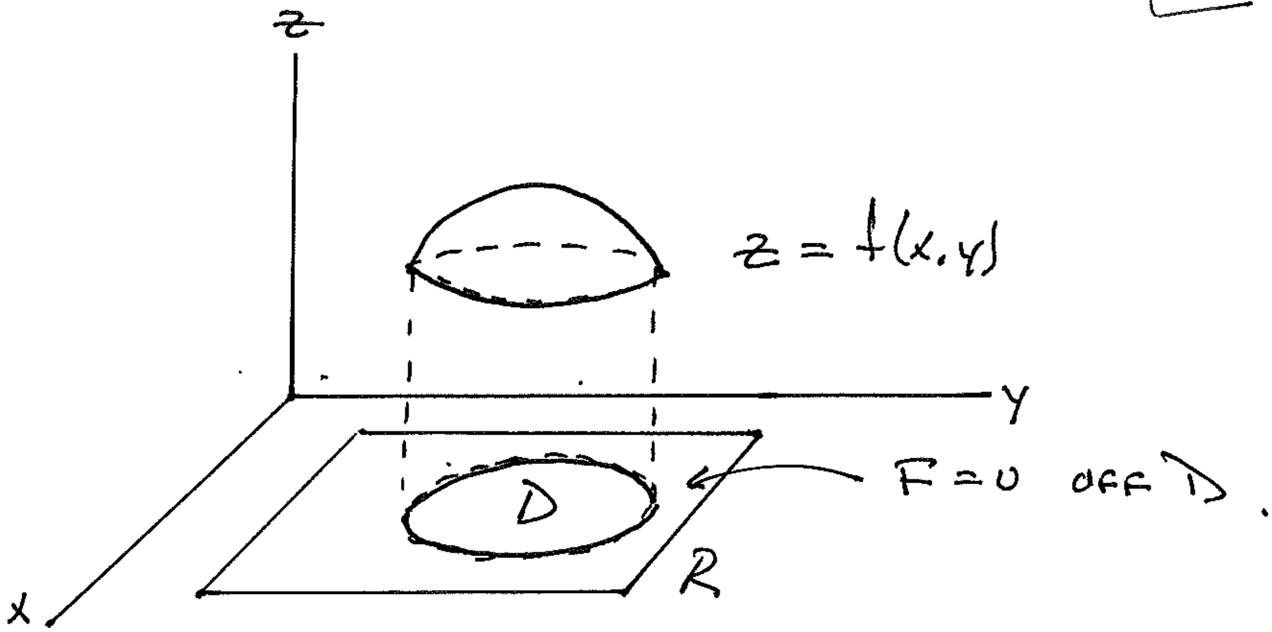


Define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R - D \end{cases}$$

Then define

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA .$$

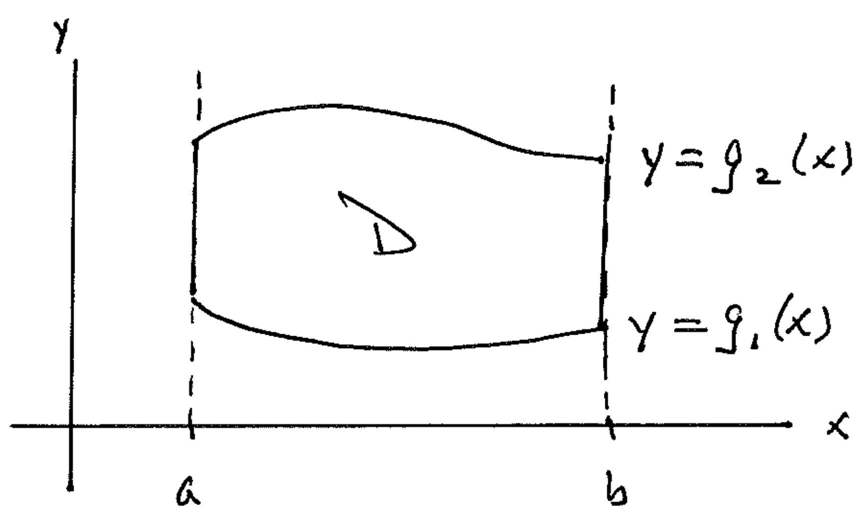


TO EVALUATE $\iint_D f\,dA$ WE DISTINGUISH BETWEEN TWO TYPES OF REGIONS.

TYPE I:

$$D = \{(x,y) \mid a \le x \le b \text{ and } g_1(x) \le y \le g_2(y)\}$$

WHERE g_1, g_2 ARE CONTINUOUS ON $[a,b]$.



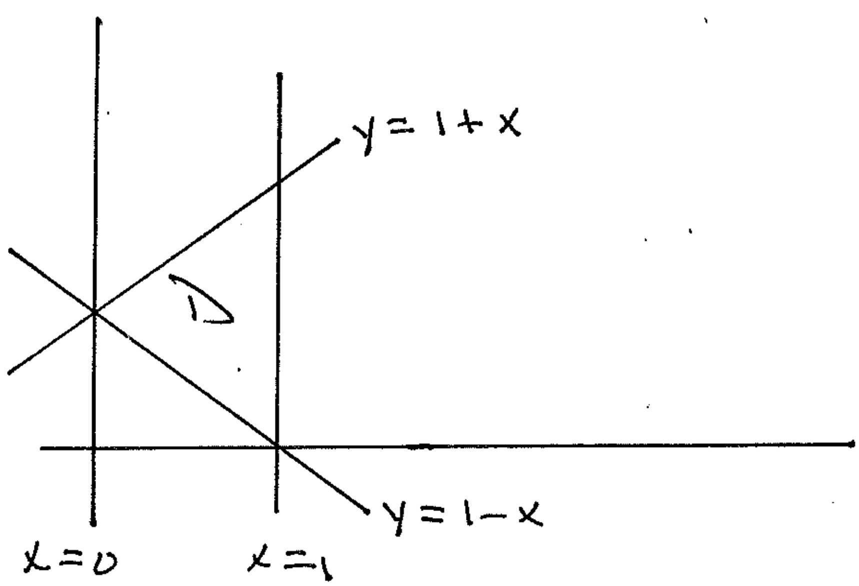
IN THIS CASE

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

i.e. INTEGRATE WRT y FROM $y = g_1(x)$ TO $y = g_2(x)$, THEN INTEGRATE THE RESULTING FUNCTION OF x FROM $x = a$ TO $x = b$.

Ex. LET D BE THE REGION BOUNDED BY

$$D: \begin{cases} 0 \le x \le 1 \\ 1 - x \le y \le 1 + x \end{cases}$$



Find $\iint_D xy\, dA$.

$$\int_0^1 \int_{1-x}^{1+x} xy\, dy\, dx = \int_0^1 \frac{1}{2} x y^2 \Big|_{1-x}^{1+x} dx$$

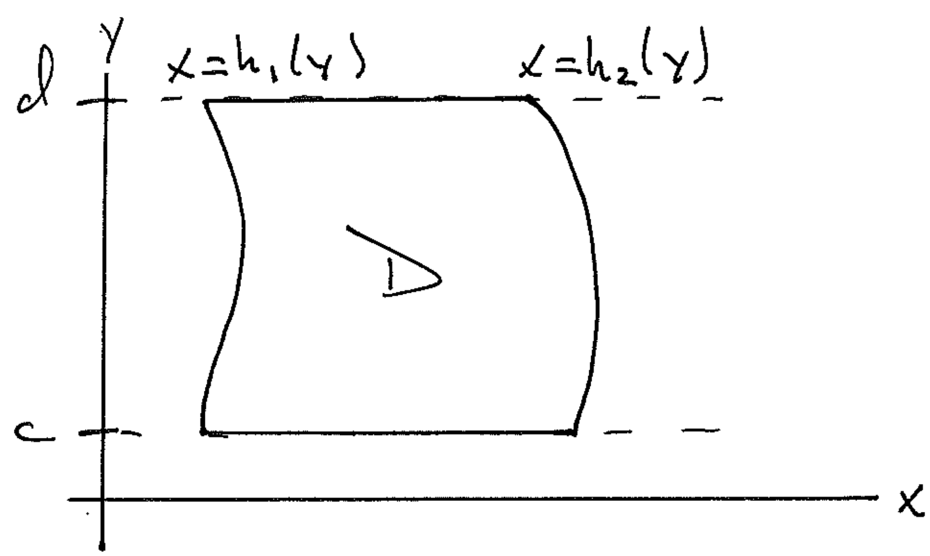
$$= \int_0^1 \frac{1}{2} x\left((1+x)^2 - (1-x)^2\right) dx$$

$$= \int_0^1 \frac{1}{\cancel{2}} x \cdot 2x \cdot \cancel{2}\, dx = 2\int_0^1 x^2\, dx$$

$$= \frac{2}{3} x^3 \Big|_0^1 = \boxed{\frac{2}{3}}$$

Type II.

$$D = \{(x,y) \mid c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$$



IN THIS CASE:

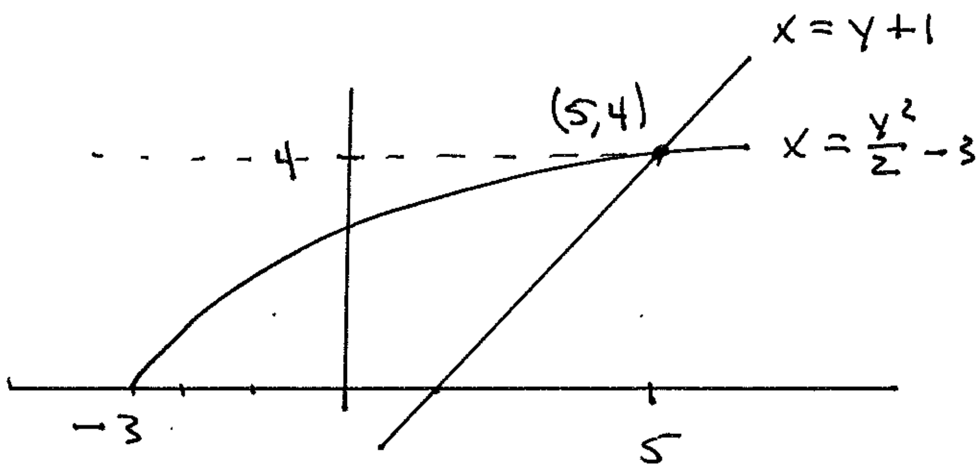
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Ex. LET D BE BOUNDED BY

$$D: \begin{cases} 0 \le y \le 4 \\ \frac{y^2}{2} - 3 \le x \le y + 1 \end{cases}$$

FIND $\iint_D xy\,dA$.

$$\iint_D xy\,dA = \int_0^4 \int_{\frac{y^2}{2}-3}^{y+1} xy\,dx\,dy$$



$$= \int_0^4 \frac{1}{2}x^2 y \Big|_{\frac{y^2}{2}-3}^{y+1} dy$$

$$= \int_0^4 \frac{1}{2}y\left[-\frac{1}{4}y^4 + 4y^2 + 2y - 8\right] dy$$

$$= \int_0^4 \left(-\frac{1}{8}y^5 + 2y^3 + y^2 - 4y\right) dy$$

$$= -\frac{1}{48}y^6 + \frac{1}{2}y^4 + \frac{1}{3}y^3 - 2y^2 \Big|_0^4$$

$$= -\frac{4^6}{4^2 \cdot 3} + \frac{4^4}{2} + \frac{4^3}{3} - 2 \cdot 4^2$$

$$= -\frac{256}{3} + 128 + \frac{64}{3} - 32 = \boxed{32}$$

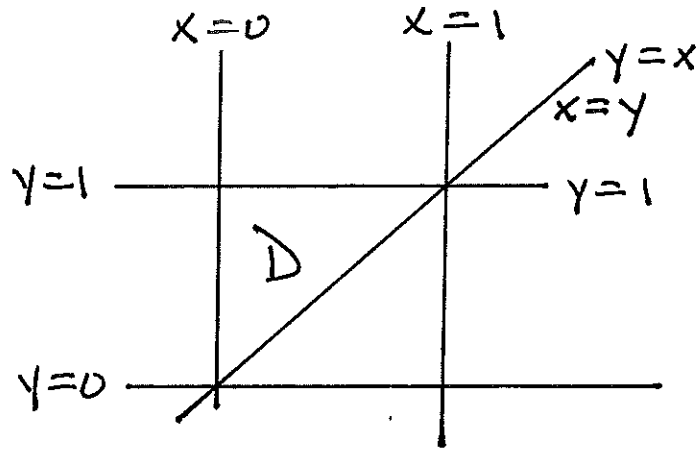
SOMETIMES A REGION IS OF BOTH TYPES SIMULTANEOUSLY. IN THIS CASE ONE CAN CHANGE THE ORDER OF INTEGRATION BY FIRST DRAWING A CAREFUL PICTURE OF D AND RE-WRITING THE INEQUALITIES THAT DEFINE IT.

EX. EVALUATE $\displaystyle\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$

TYPE I: $D: \begin{cases} 0 \le x \le 1 \\ x \le y \le 1 \end{cases}$

OR

TYPE II: $D: \begin{cases} 0 \le y \le 1 \\ 0 \le x \le y \end{cases}$



THUS

$$\int_0^1 \int_x^1 e^{x/y}\, dy\, dx = \int_0^1 \int_0^y e^{x/y}\, dx\, dy$$

$$= \int_0^1 y e^{x/y} \Big|_{x=0}^{x=y}\, dy$$

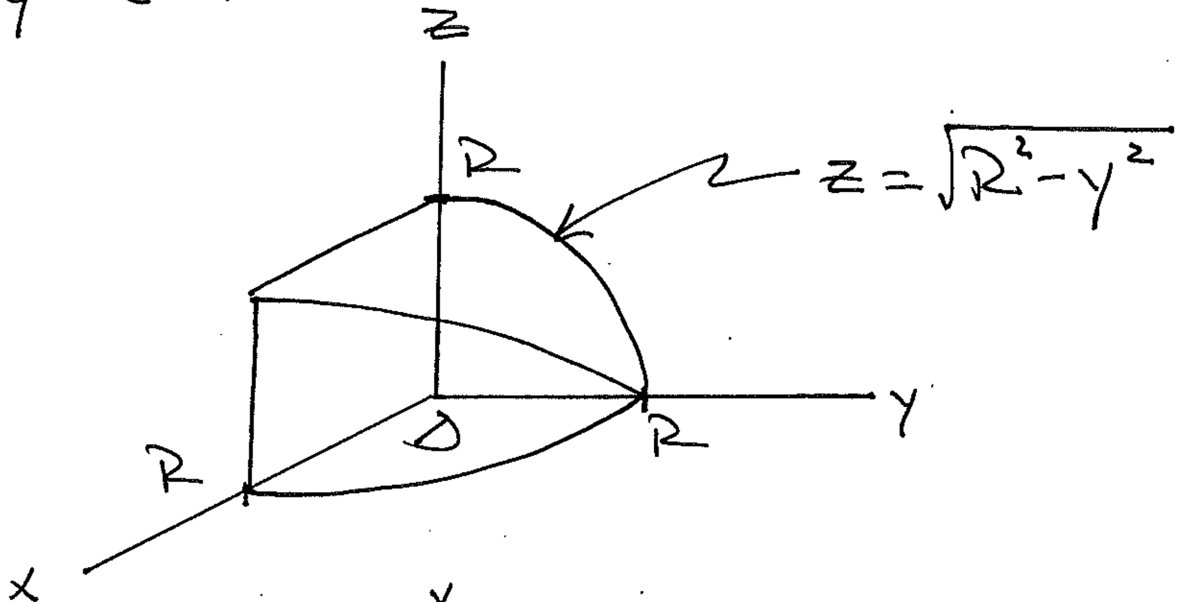
$$= \int_0^1 y \cdot (e-1)\, dy = \frac{1}{2}(e-1)y^2 \Big|_0^1$$

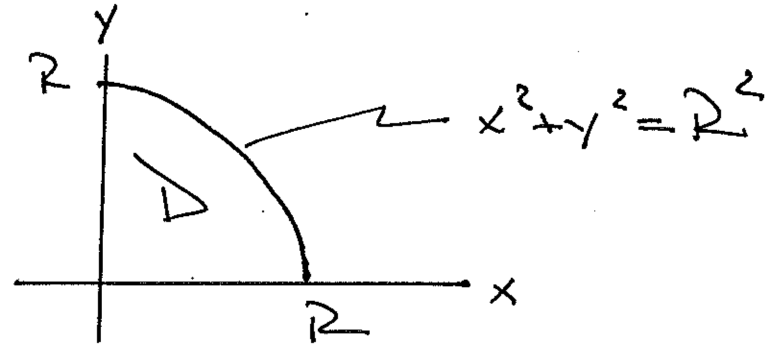
$$= \boxed{\frac{e-1}{2}}.$$

EX. FIND THE VOLUME OF THE REGION BOUNDED BY THE CYLINDERS

$$x^2+y^2=R^2 \quad \text{AND} \quad y^2+z^2=R^2$$

WE COMPUTE THE VOLUME IN THE 1^{ST} OCTANT AND MULTIPLY BY 8 :





$$V=8\int_0^R\int_0^{\sqrt{R^2-x^2}}\sqrt{R^2-y^2}\,dy\,dx=\int_0^R\int_0^{\sqrt{R^2-y^2}}\sqrt{R^2-y^2}\,dx\,dy$$

$$= 8\int_0^R \sqrt{R^2-y^2} \cdot x \Big|_{x=0}^{x=\sqrt{R^2-y^2}} dy$$

$$= 8\int_0^R (R^2-y^2)\, dy$$

$$= 8\left(R^2 y - \tfrac{1}{3}y^3\right)\Big|_0^R$$

$$= 8\left(R^3 - \tfrac{1}{3}R^3\right) - 0 = \boxed{\frac{16}{3} R^3}$$