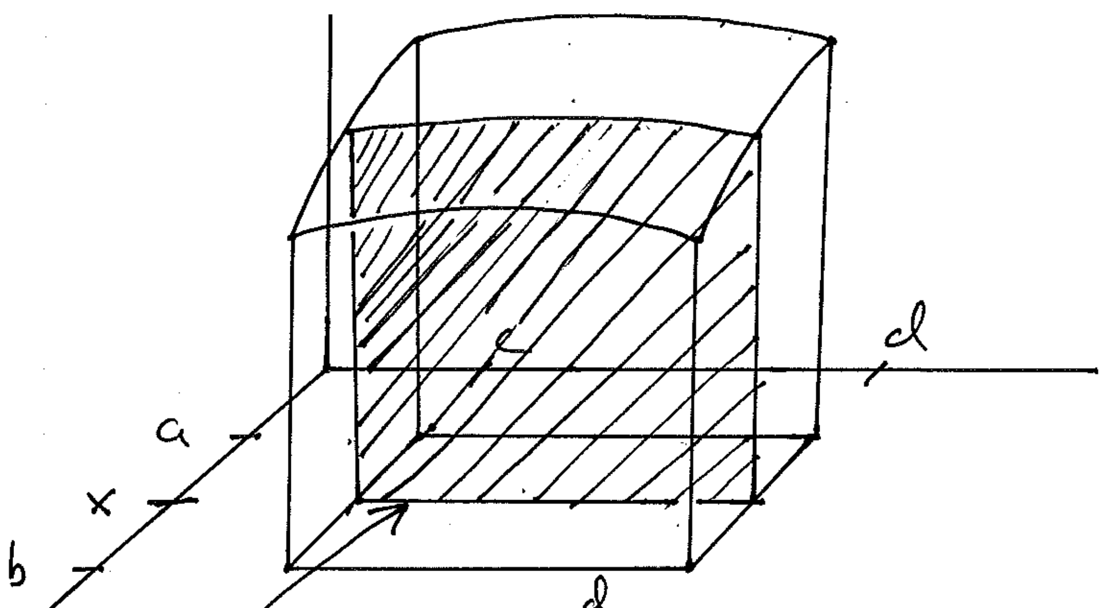
## (15.2) ITERATED INTEGRALS

LET $f(x,y)$ BE INTEGRABLE ON $R = [a,b] \times [c,d]$. DEFINE A FUNCTION

$$A(x) = \int_c^d f(x,y)\,dy \qquad (a \le x \le b)$$

i.e. HOLD $x$ FIXED IN $[a,b]$ AND INTEGRATE $f(x,y)$ WITH RESPECT TO $y$ FROM $y=c$ TO $y=d$. THIS PROCESS COULD BE CALLED PARTIAL INTEGRATION WITH RESPECT TO $y$.




$$A(x) = \int_c^d f(x,y)\,dy$$

= AREA OF PLANE SLICE

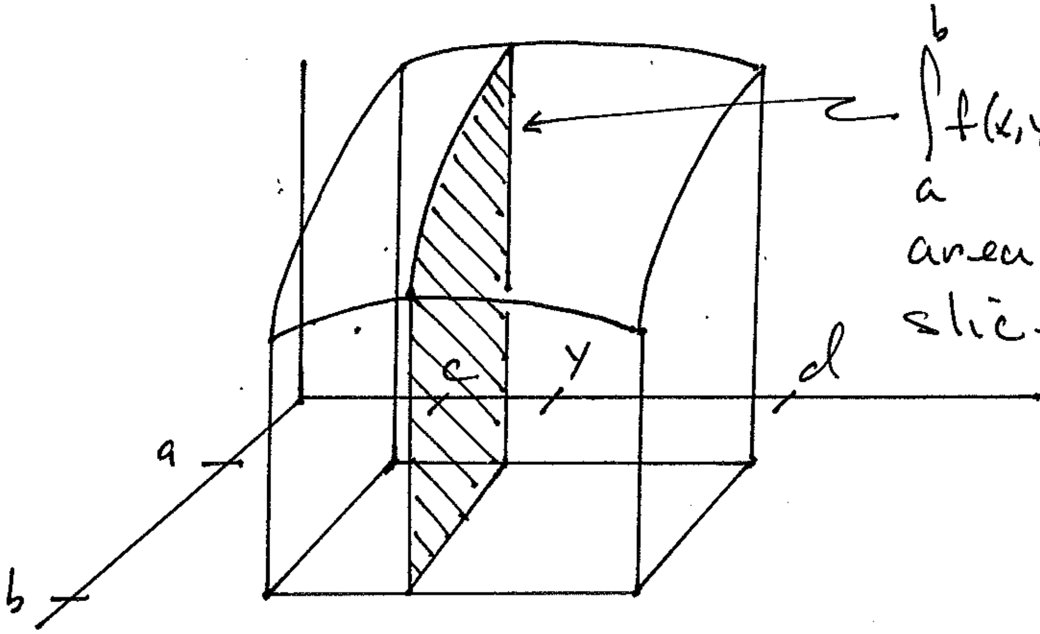
NOW INTEGRATE $A(x)$ FROM $x=a$ TO $x=b$ TO GET

$$\int_a^b A(x)\,dx = \int_a^b \left( \int_c^d f(x,y)\,dy \right) dx$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

WE CALL THIS AN ITERATED INTEGRAL OF $f$ OVER $R$. WE GET ANOTHER SUCH INTEGRAL BY DOING THINGS IN THE OPPOSITE ORDER

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left( \int_a^b f(x,y)\,dx \right) dy$$




$\int_a^b f(x,y)\,dx =$ area of planar slice.

THEOREM (FUBINI)

If $f(x,y)$ is CONTINUOUS ON $R=[a,b]\times[c,d]$, THEN

$$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx$$

$$=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

EX. LET $f(x,y)=2x+3y$ AND $R=[1,2]\times[1,3]$. THEN

$$\int_1^2\int_1^3 (2x+3y)\,dy\,dx=\int_1^2 \left(2xy+\tfrac{3}{2}y^2\right)\Big|_1^3\,dx$$

$$=\int_1^2\left[\left(6x+\tfrac{27}{2}\right)-\left(2x+\tfrac{3}{2}\right)\right]dx$$

$$=\int_1^2 (4x+12)\,dx=(2x^2+12x)\Big|_1^2$$

$$=(8+24)-(2+12)=\boxed{18}$$

CHECK $\int_1^3\int_1^2 (2x+3y)\,dx\,dy=18$ ALSO.

<u>Ex.</u> $R = [0,3] \times [0,1]$.

$$\iint_R (6x^2y^3 - 5y^4)\, dA$$

SOMETIMES THERE IS A PREFERRED ORDER OF INTEGRATION

<u>Ex.</u> $R = [0,1] \times [0,1]$

$$\iint_R \frac{x}{1+xy}\, dA = \int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx$$

$$= \ldots = \boxed{2\ln 2 - 1}$$

TRY OTHER ORDER:

$$\int_0^1 \int_0^1 \frac{x}{1+xy}\, dx\, dy = \ldots.$$

THIS IS DOABLE, BUT MORE DIFFICULT.

Ex. $R = [1,2] \times [0,5]$ , $f(x,y) = x^2 y$

find $f_{avg} = \frac{1}{A(R)} \iint_R x^2 y \, dA$ .

$$A(R) = 1 \cdot 5 = 5$$

$$\iint_R x^2 y \, dA = \int_1^2 \int_0^5 x^2 y \, dy \, dx = \frac{1}{2} \int_1^2 x^2 y^2 \Big|_0^5 \, dx$$

$$= \frac{25}{2} \int_1^2 x^2 \, dx = \frac{25}{2} \cdot \frac{1}{3} x^3 \Big|_1^2$$

$$= \frac{25}{6}(8-1) = \frac{25 \cdot 7}{6}$$

$$f_{avg} = \frac{1}{5} \cdot \frac{25 \cdot 7}{6} = \frac{5 \cdot 7}{6} = \boxed{\frac{35}{6}} .$$