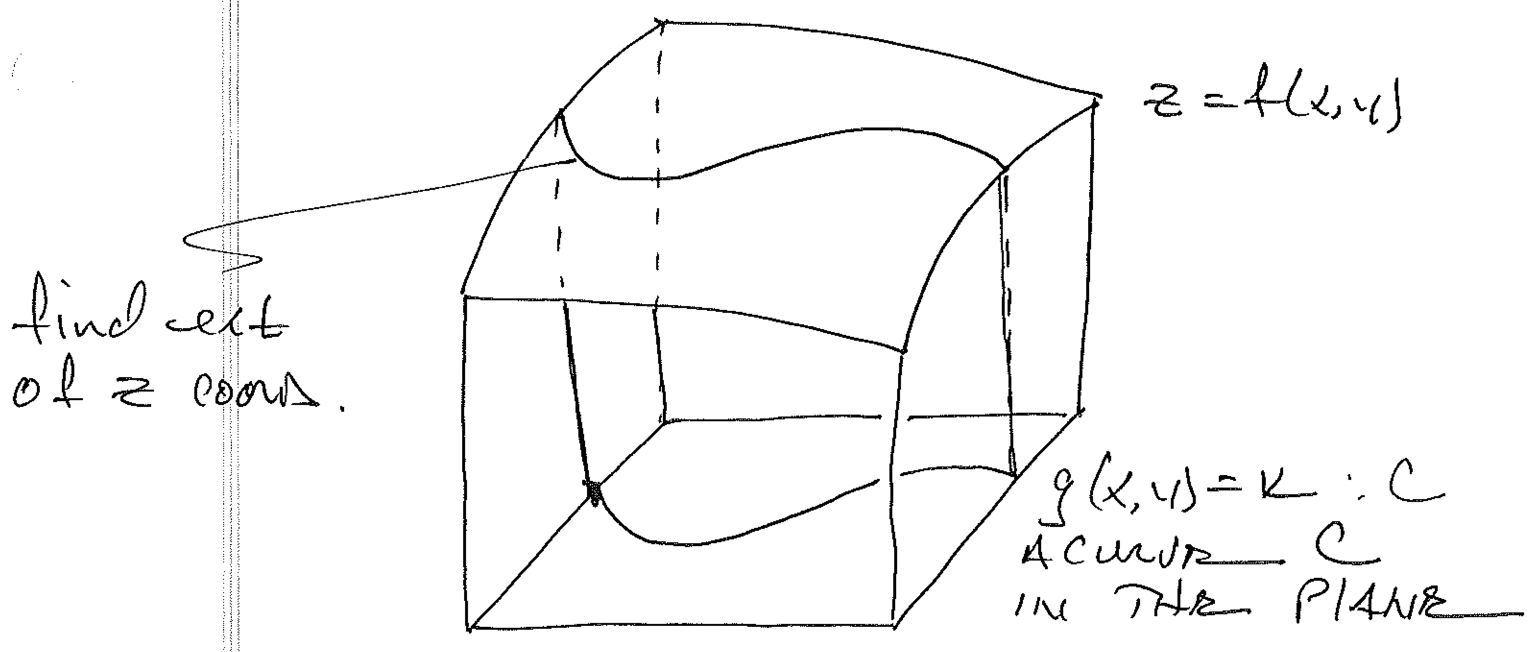
(14.8) LAGRANGE MULTIPLIERS

CONSTRAINED EXTREMA:

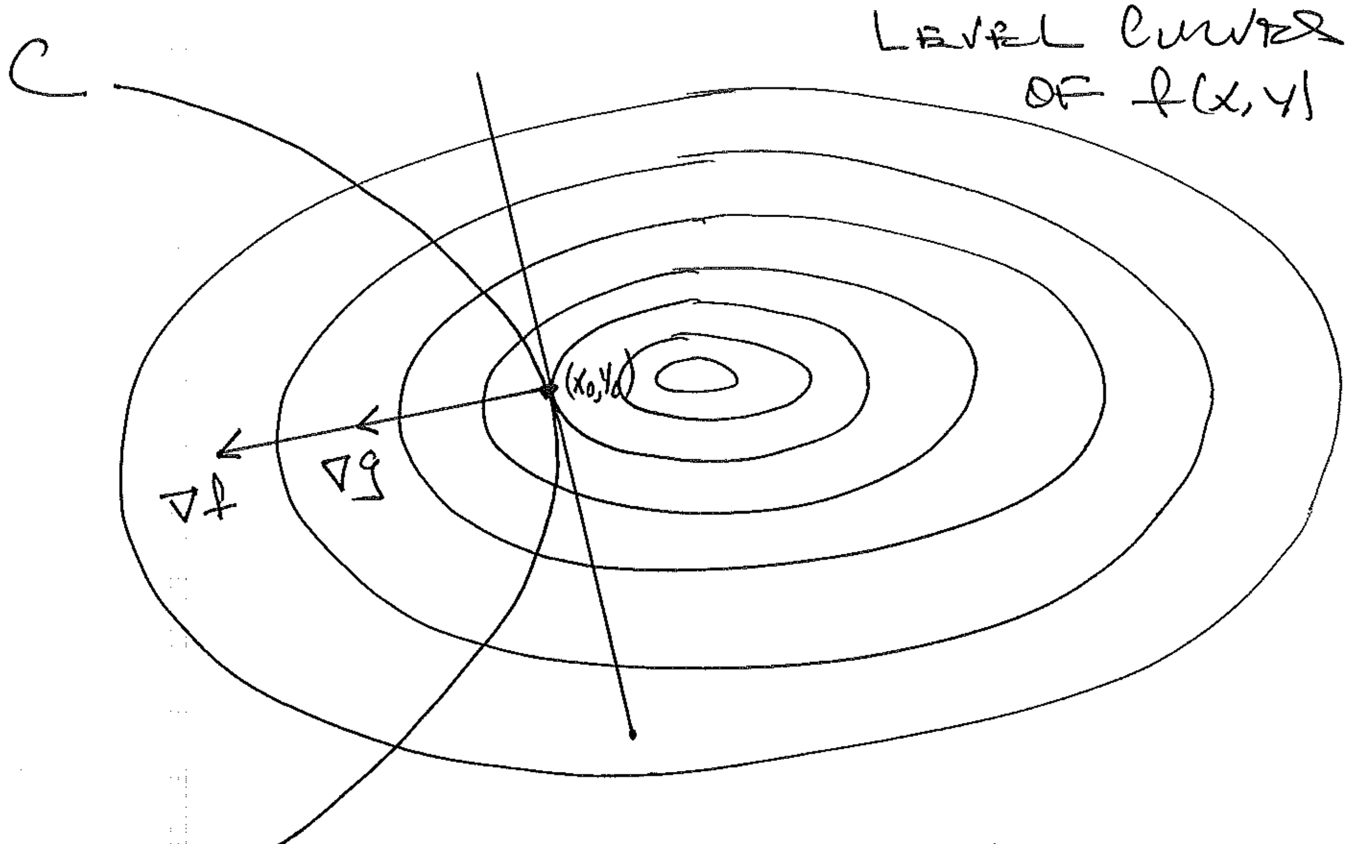
OUR GOAL IN THIS SECTION IS TO SOLVE PROBLEMS OF THE FOLLOWING TYPE.

- FIND THE EXTREME VALUES OF $f(x,y)$ SUBJECT TO THE CONSTRAINT $g(x,y) = K$ = CONST.



- FIND EXTREME VALUES OF $F(x,y,z)$ SUBJECT TO CONSTRAINT $G(x,y,z) = K$
- FIND EXTREMA OF $F(x,y,z)$ SUBJECT TO SEVERAL CONSTRAINTS: $G_1(x,y,z) = K_1$ AND $G_2(x,y,z) = K_2$.

2-dim PROBLEM



CONSTRAINT CURVE $g(x,y)=k$

WE SEE THAT THE EXTREMA OF $f(x,y)$ ALONG C WILL OCCUR AT A POINT (x_0, y_0) WHERE BOTH C AND A LEVEL CURVE OF f HAVE THE SAME TANGENT LINE

i.e. C AND A level curve OF f HAVE NORMAL VECTORS THAT ARE PARALLEL.

i.e.

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

FOR SOME $\lambda \in \mathbb{R}$, CALLED A <u>LAGRANGE MULTIPLIER</u>.

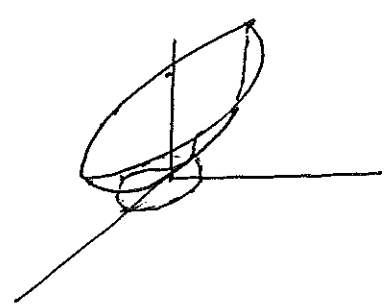
THUS WE SEEK SOLUTIONS (x, y, λ) TO THE EQNS

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = k \end{cases}$$

THESE ARE 3 EQNS IN THE 3 UNKNOWNS (x, y, λ). WE THEN CHECK THE VALUE $f(x,y)$ AT THESE POINTS TO DETERMINE MAXIMA & MINIMA.

<u>EX</u> FIND EXTREME VALUES OF $f(x,y) = x^2 + 2y^2$ SUBJECT TO CONSTRAINT

$$x^2 + y^2 = 1.$$



(RECALL WE DID THIS PROBLEM IN THE LAST SECTION BY FINDING EXTREMA OF

$$h(t) = f(\cos t, \sin t) = 1 + \sin^2 t$$

ON $0 \le t \le 2\pi$. WE FOUND THAT

MINIMA : $f(1,0) = f(-1,0) = 1$
MAXIMA : $f(0,1) = f(0,-1) = 2$.)

USING THE METHOD OF LAGRANGE MULTIPLIERS WE HAVE

$$\begin{cases} 2x = \lambda \cdot 2x \\ 4y = \lambda \cdot 2y \\ x^2 + y^2 = 1 \end{cases}$$

i.e.

$$\begin{cases} (\lambda - 1)x = 0 \\ (\lambda - 2)y = 0 \\ x^2 + y^2 = 1 \end{cases}$$

$\lambda = 1 \Rightarrow y = 0 \rightarrow x = \pm 1 \Rightarrow (1,0), (-1,0)$
$\lambda = 2 \rightarrow x = 0 \rightarrow y = \pm 1 \rightarrow (0,1), (0,-1)$

AND AS WE'VE SEEN

$f(1,0) = f(-1,0) = 1$ is min.
$f(0,1) = f(0,-1) = 2$ is max.

THE SAME PROCEDURE WORKS FOR A FUNCTION $f(x,y,z)$ OF 3 VARIABLES, SUBJECT TO $g(x,y,z)=K$

i.e. WE SOLVE

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = K \end{cases}$$

FOR ALL (x,y,z,λ), THEN EVALUATE $f(x,y,z)$ TO DETERMINE max & min VALUES.

WE REQUIRE IN BOTH CASES (2 VAR OR 3 VAR) THAT $\nabla g \neq 0$ OTHERWISE WE ARE AT A CRITICAL POINT OF g. IN SUCH CASES A THE CONSTRAINT CURVE MAY DEGENERATE TO A SINGLE POINT.

Ex. $f(x,y,z) = xyz$
$g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6$

$$\begin{cases} yz = \lambda \cdot 2x \\ xz = \lambda \cdot 4y \\ xy = \lambda \cdot 6z \\ x^2 + 2y^2 + 3z^2 = 6 \end{cases}$$

WE CANNOT HAVE ALL $x, y, z = 0$ (OTHERWISE $0 = 6$) .. IF SAY $y = z = 0$ THEN $x \neq 0$ & $\lambda = 0$ AND $x = \pm\sqrt{6}$

- IF $y = x = 0$ THEN $\lambda = 0$ AND $z = \pm\sqrt{2}$
- IF $x = z = 0$ THEN $\lambda = 0$ AND $y = \pm\sqrt{3}$

WE CANNOT HAVE JUST ONE $= 0$, for SAY IF $x = 0$ THEN $yz = 0$, SO EITHER $y = 0$ OR $z = 0$.

THUS WE HAVE 6 SOLUTIONS

$(\pm\sqrt{6}, 0, 0)$
$(0, \pm\sqrt{3}, 0)$
$(0, 0, \pm\sqrt{2})$

THE VALUE OF f AT EACH OF THESE POINTS IS $\boxed{0}$

NOW ASSUME NONE OF x, y, OR z ARE 0. THEN $\lambda \neq 0$!

$$x = \frac{yz}{2\lambda} \rightarrow \begin{cases} \frac{yz}{2\lambda} \cdot z = 4\lambda \cdot y \\ \frac{yz}{2\lambda} \cdot y = 6\lambda \cdot z \end{cases}$$

$$\therefore \begin{cases} z^2 = 8\lambda^2 \\ y^2 = 12\lambda^2 \end{cases} \rightarrow x^2 = \frac{8\lambda^2 \cdot 12\lambda^2}{4\lambda^2} = 24\lambda^2$$

$$\therefore \quad 24\lambda^2 + 24\lambda^2 + 24\lambda^2 = 6$$

$$\therefore 3 \cdot 6 \cdot 4 \cdot \lambda^2 = 6$$

$$\lambda^2 = \frac{1}{12}$$

$$\therefore \quad x^2 = 2 \qquad x = \pm\sqrt{2}$$

$$y^2 = 1 \qquad y = \pm\sqrt{1}$$

$$z^2 = \frac{8}{12} = \frac{2}{3} \qquad z = \pm\sqrt{\frac{2}{3}}$$

8 PTS: $\left(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}}\right) \quad f = \pm\frac{2}{\sqrt{3}}$

$$\left.\begin{matrix} +++ \\ +-- \\ -+- \\ --+ \end{matrix}\right\} \xrightarrow{f} +\frac{2}{\sqrt{3}} \quad \text{max}$$

$$\left.\begin{matrix} -++ \\ +-+ \\ ++- \\ --- \end{matrix}\right\} \xrightarrow{f} -\frac{2}{\sqrt{3}} \quad \text{min}.$$

Ex. $f(x,y,z) = 2x + 6y + 10z$

$g(x,y,z) = x^2 + y^2 + z^2 = 35$

$$\begin{cases} 2 = \lambda \cdot 2x \\ 6 = \lambda \cdot 2y \\ 10 = \lambda \cdot 2z \end{cases} \rightarrow \quad \begin{aligned} x &= \tfrac{1}{\lambda} \\ y &= 3/\lambda \\ z &= 5/\lambda \end{aligned}$$

$$\frac{1}{\lambda^2} + \frac{9}{\lambda^2} + \frac{25}{\lambda^2} = 35 \longrightarrow \lambda^2 = 1$$
$$\longrightarrow \lambda = \pm 1$$

2 PTS :

$f(1, 3, 5) =$

$f(-1, -3, -5) =$

FOR PROBLEMS INVOLVING 2 CONSTRAINTS SUCH AS

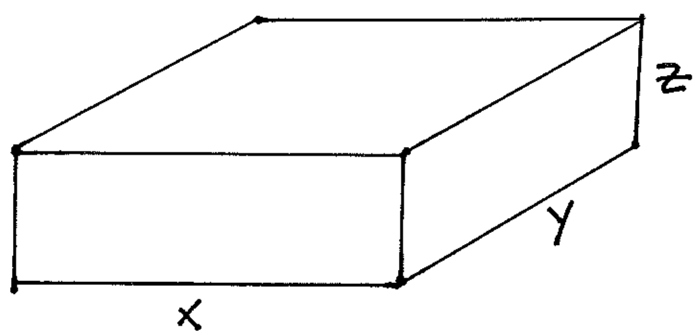
EXTREMA OF : $f(x,y,z)$

SUBJECT TO : $\begin{cases} g(x,y,z) = K_1 \\ h(x,y,z) = K_2 \end{cases}$

WE REQUIRE $\nabla f = \lambda \nabla g + \mu \nabla h$

i.e. $$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g = k_1 \\ h = k_2 \end{cases}$$

<u>EX.</u> FIND THE max & min Volume OF A RECTANGULAR BOX WHOSE SURFACE AREA IS 1500 cm², AND WHOSE TOTAL EDGE LENGTH IS 200 cm.



$$V = xyz$$

$$S = 2(xy + yz + xz) = 1500$$
$$L = 4(x + y + z) = 200$$