(14.7) MAXIMA & MINIMA

DEFN:
A FCN $f(x,y)$ HAS A <u>LOCAL MAXIMUM</u> AT (a,b) IF

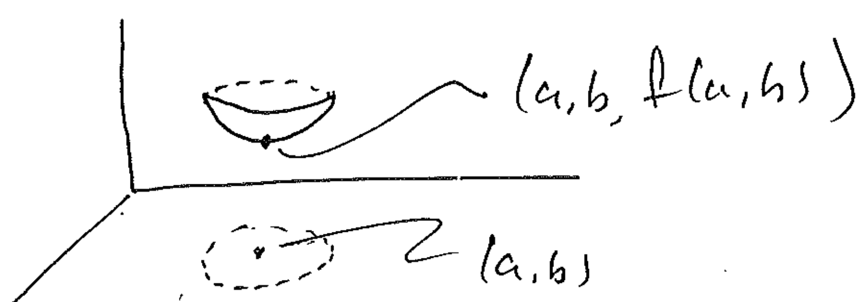
$$f(x,y) \le f(a,b)$$

FOR ALL (x,y) IN SOME DISK CENTERED AT (a,b).

SIMILARLY, $f(x,y)$ HAS A <u>LOCAL MINIMUM</u> AT (a,b) IF

$$f(x,y) \ge f(a,b)$$

for ALL $(x,y) \in D$



IF THE INEQUALITIES HOLD FOR ALL $(x,y) \in Dom(f)$, WE CALL (a,b) AN <u>ABSOLUTE</u> MAXIMUM OR MINIMUM.

THM

IF f HAS A LOCAL EXTREMUM (i.e. MAXIMUM OR MINIMUM) AT (a,b), AND $f_x(a,b)$ AND $f_y(a,b)$ EXIST, THEN

$$f_x(a,b) = 0 = f_y(a,b) \; .$$

PROOF:

LET $g(x) = f(x,b)$. THEN $g(x)$ HAS A LOCAL EXTREMUM AT $x = a$.

$\therefore \quad 0 = g'(a) = f_x(a,b)$ BY A THM. FROM 11a 19a.

SIMILARLY $h(y) = f(a,y)$ HAS A LOCAL EXTREMUM AT $y = b$ SO

$$0 = h'(b) = f_y(a,b) \; .$$

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DEFN:

WE CALL (a,b) A <u>CRITICAL POINT</u> OF $f(x,y)$ IF EITHER

(i) $f_x(a,b) = 0 = f_y(a,b)$

OR (ii) ONE OR BOTH OF f_x, f_y FAIL TO EXIST AT (a,b)

THUS THE THM SAYS THAT

- IF (a,b) IS A LOCAL EXTREMUM, THEN (a,b) IS A CRITICAL POINT

THE CONVERSE IS FALSE (AS IN 11a/19a).

<u>Ex.</u> $f(x;y) = x^2+2y^2-8x-12y+39$.

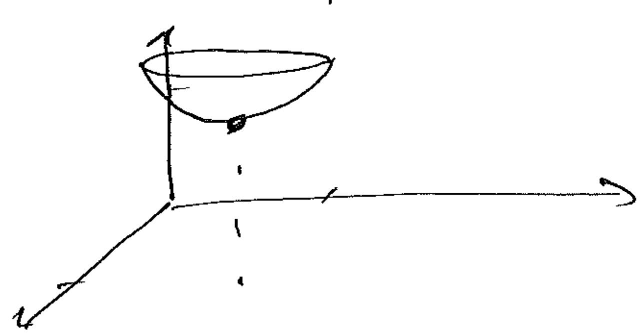
THE CRITICAL PTS. OF f ARE GIVEN BY:

$f'_x = 2x-8 = 0 \longrightarrow x=4$ $\quad\boxed{(4,3)}$
$f_y = 4y-12 = 0 \longrightarrow y=3$

OBSERVE THAT UPON COMP. THE SQ. WE HAVE

$$f(x,y) = 5+(x-4)^2+2(y-3)^2$$

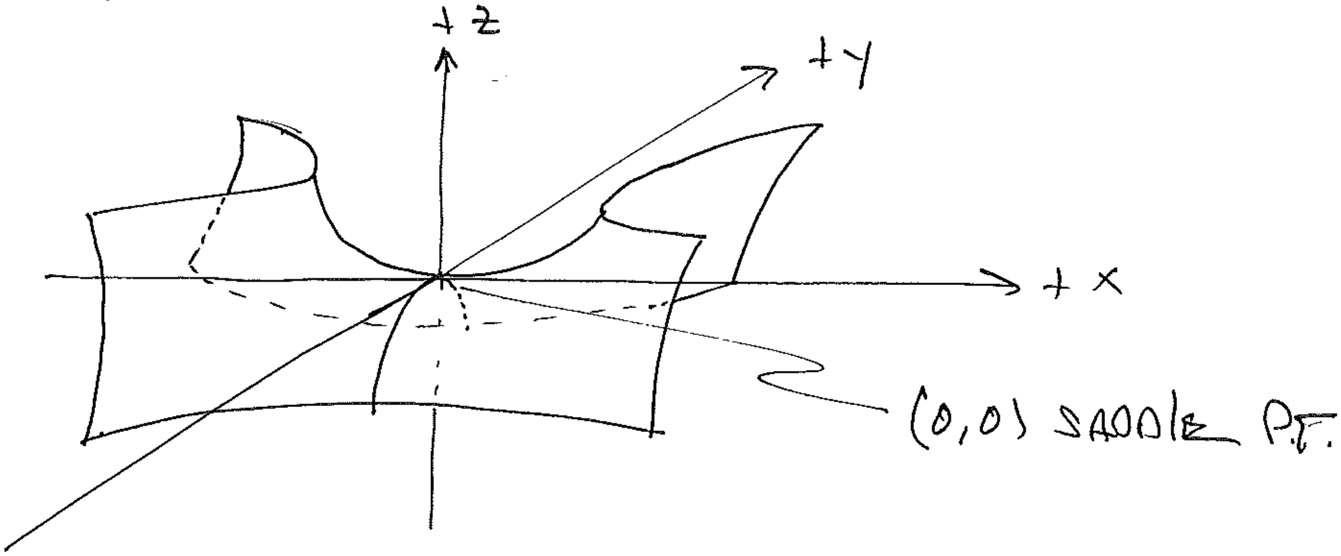
$\therefore$ f HAS A LOCAL (INFACT ABSOLUTE) MIN AT $(4,3)$



Ex. $f(x,y) = x^2 - y^2$

$$\left.\begin{aligned} f_x &= 2x = 0 \\ f_y &= -2y = 0 \end{aligned}\right\} \longrightarrow \begin{gathered}\text{CRITICAL PT.} \\ (0,0)\end{gathered}$$

BUT f HAS NO MIN OR MAX AT $(0,0)$



SO.. A CRITICAL PT. NEED NOT BE AN EXTREMA, EVEN IF BOTH DERIVATIVES EXIST AND ARE CONTINUOUS.

THM (2ND DERIVATIVE TEST)
SUPPOSE THE 2ND PARTIAL DERIVATIVES OF f ARE CONT. IN A DISK ABOUT (a,b), where $f_x(a,b)=0=f_y(a,b)$.
LET

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a,b)\cdot f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2$$

THEN

(a) If $D>0$ AND $f_{xx}(a,b)>0$ THEN f HAS A LOCAL <u>min</u> AT (a,b)
(b) If $D>0$ AND $f_{xx}(a,b)<0$ THEN f HAS A <u>local max</u> AT (a,b).

(c) If $D<0$ THEN $f(a,b)$ is <u>NOT</u> A local EXTREMUM, AND IN FACT (a,b) is A SADDLE POINT.

<u>RMKS</u>

- If $D=0$, WE CONCLUDE NOTHING!
- $D>0 \Rightarrow f_{xx}\cdot f_{yy} > f_{xy}^2 \geq 0$
  $\Rightarrow f_{xx}$ & f_{yy} NON-ZERO & SAME SIGN
  $\therefore$ COULD TEST f_{yy} IN (a) & (b).

Ex Find and classify the crit. pts. of

$$f(x,y) = 3x - x^3 - 2y^2 + y^4.$$

$$f_x = 3 - 3x^2 = -3(x-1)(x+1) = 0$$
$$f_y = -4y + 4y^3 = +4y(y-1)(y+1) = 0$$

Critical pts.: $(1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1)$

$$f_{xx} = -6x$$
$$f_{yy} = -4 + 12y^2$$
$$f_{xy} = 0$$
$$D = (-6x)(12y^2-4) = -24 \cdot x\,(3y^2-1)$$

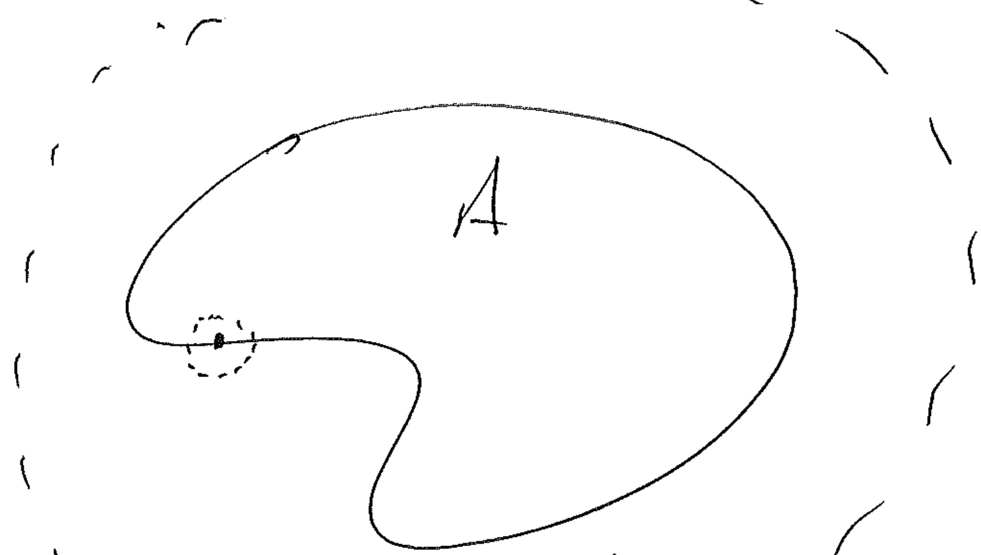
	D	f_{xx}	case
$(1,0)$	$+$	$-$	b. max
$(1,1)$	$-$	$-$	c. saddle
$(1,-1)$	$-$	$-$	c. saddle
$(-1,0)$	$-$	$+$	c. saddle
$(-1,1)$	$+$	$+$	a. min
$(-1,-1)$	$+$	$+$	a. min

To find extrema of $f(x,y)$ on a closed bounded region of $\mathbb{R}^2$ we must also check boundary points.

Thm

If $f(x,y)$ is continuous on a closed bounded set $A \subseteq \mathbb{R}^2$ then f attains both an abs. max & abs min at some points $(x_1, y_1), (x_2, y_2) \in A$:

$$\begin{cases} \max & f(x_1, y_1) \text{ at } (x_1, y_1) \\ \min & f(x_2, y_2) \text{ at } (x_2, y_2). \end{cases}$$



Bounded: contained in some disk

Closed: includes boundary points.

Boundary pt: any disk about pt contains pts in & out of A.

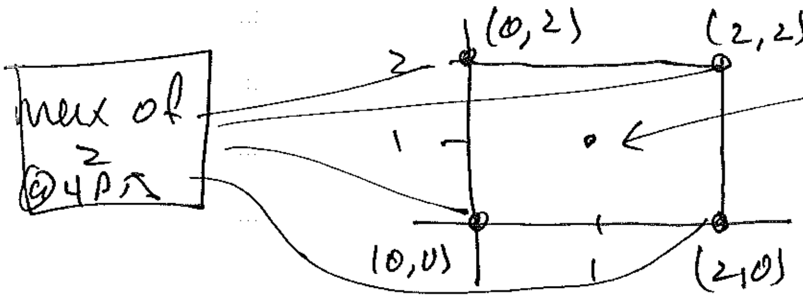
SO TO FIND EXTREMA ON A CLOSED BDD SET A:

(1) FIND CRITICAL POINTS & CRITICAL VALUES IN A

(2) find EXTREMA ON BOUNDARY OF A

(3) COMPARE EXTREMA found in (1) & (2)

<u>Ex.</u> $f(x,y) = (x-1)^2 + (y-1)^2$ ON RECTANGLE $[0,2] \times [0,2]$



ABS min OF 0 AT (1,1) CRIT. PT.

<u>BOUNDARY</u>

$x = 0 : f(0,y) = (y-1)^2 + 1$

$x = 2 : f(2,y) = (y-1)^2 + 1$

$y = 0 : f(x,0) = (x-1)^2 + 1$

$y = 2 : f(x,2) = (x-1)^2 + 1$

min	max
1 @ $y=1$	2 @ $y=0$, $y=2$
"	"
1 @ $x=1$	2 @ $x=0$, $x=2$
"	"

ABS min $f(1,1) = 0$

ABS max $f(0,0) = f(0,2) = f(2,0) = f(2,2) = 2$

EX FIND EXTREMA OF

$$f(x,y) = x^2 + 2y^2$$

ON THE CLOSED DISC $x^2+y^2 \le 1$.

CRIT. PTS. $\left.\begin{array}{l} f_x = 2x = 0 \\ f_y = 4y = 0 \end{array}\right\} \to (0,0)$

$\left.\begin{array}{l} f_{xx} = 2 \\ f_{yy} = 4 \\ f_{xy} = 0 \end{array}\right\} D = 2\cdot 4 - 0^2 = 8 > 0$

$\therefore f(0,0) = 0$ is a local min

CHECK BOUNDARY: $C \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t \le 2\pi$

$$h(t) = f(\cos t, \sin t) = \cos^2 t + 2\sin^2 t$$
$$= 1 + \sin^2 t$$

$h'(t) = 2\sin t\cos t = 0 \to \begin{cases} t = 0, \pi \\ t = \pi/2, 3\pi/2 \end{cases}$

$\to (x,y) = \begin{cases} (1,0), (-1,0) \\ (0,1), (0,-1) \end{cases}$

$f(1,0) = f(-1,0) = 1^2 + 0 = 1$

$f(0,1) = f(0,-1) = 0 + 2\cdot 1^2 = 2$ max.

$\therefore$ ABS min: $f(0,0) = 0$

ABS max: $f(0,-1) = f(0,1) = 2$