(14.5) CHAIN RULE

THM (, 1ST VERSION)
LET $z = f(x,y)$ BE COMPOSED WITH TWO FUNCTIONS

$$x = g(t), \quad y = h(t)$$

THEN

$$z = f\big(g(t), h(t)\big)$$

IS ITSELF A FCN. OF t.
ITS DERIVATIVE IS

$$\frac{dz}{dt} = f_x\big(g(t), h(t)\big)\, g'(t) + f_y\big(g(t), h(t)\big) \cdot h'(t)$$

i.e.

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

PROVIDED f, g, h ARE ALL DIFFERENTIABLE FCNS.
(SEE PF. ON P. 901)

Ex: $z = f(x,y) = x^2y + 3xy^2$

$$\begin{cases} x = \cos t & \dfrac{dx}{dt} = -\sin t \\ y = \sin t & \dfrac{dy}{dt} = \cos t. \end{cases}$$

$$\frac{\partial f}{\partial x} = 2xy + 3y^2 = 2\cos t \sin t + 3\sin^2 t$$

$$\frac{\partial f}{\partial y} = x^2 + 6xy = \cos^2 t + 6\cos t \sin t$$

$\therefore$

$$\frac{dz}{dt} = (2\cos t \sin t + 3\sin^2 t)(-\sin t) + (\cos^2 t + 6\cos t \sin t)(\cos t)$$

$$= -2\cos t \sin^2 t - 3\sin^3 t + \cos^3 t + 6\cos^2 t \sin t.$$

Directly:

$z = f(\cos t, \sin t) = \cos^2 t \sin t + 3\cos t \sin^2 t.$

$$\frac{dz}{dt} = \dots\dots\dots$$

PROOF OF 1^{ST} VERSION

LET $z = f(x,y)$, $x = g(t)$, $y = h(t)$
AND SUPPOSE f, g, h ARE ALL DIFF.

LET $\Delta x = g(t+\Delta t) - g(t)$
$\Delta y = h(t+\Delta t) - h(t)$
AND

$$\Delta z = f(g(t+\Delta t), h(t+\Delta t)) - f(g(t), h(t))$$
$$= f(x+\Delta x, y+\Delta y) - f(x,y)$$

SINCE f IS DIFF. WE HAVE

$$\Delta z = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ & $\varepsilon_2 \to 0$ AS $(\Delta x, \Delta y) \to (0,0)$.

SINCE g, h DIFF (& $\therefore$ CONT) AT t, WE HAVE $\Delta x \to 0$ AND $\Delta y \to 0$ when $\Delta t \to 0$. THUS $\Delta t \to 0$ IMPLIES $\varepsilon_1 \to 0$ AND $\varepsilon_2 \to 0$.

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0}\left(f_x(x,y)\frac{\Delta x}{\Delta t} + f_y(x,y)\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}\right)$$
$$= f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt} + 0 + 0$$
$$= f_x(g(t),h(t))\cdot g'(t) + f_y(g(t),h(t))\cdot h'(t). \qquad ///$$

THM (2ND VERSION)
LET $z = f(x, y)$ BE COMPOSED WITH $x = g(s,t)$ AND $y = h(s,t)$. THEN

$$z = f(g(s,t), h(s,t))$$

HAS PARTIALS WRT s & t:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Ex. $z = e^{x+2y}$, $x = \frac{s}{t}$, $y = \frac{t}{s}$

$$\frac{\partial z}{\partial s} = e^{\left(\frac{s}{t} + \frac{2t}{s}\right)}\left(\frac{1}{t} - \frac{2t}{s^2}\right)$$

$$\frac{\partial z}{\partial t} = e^{\left(\frac{s}{t} + \frac{2t}{s}\right)}\left(-\frac{s}{t^2} + \frac{2}{s}\right)$$

Ex. $z = x \ln y$, $x = 3s + 2t$, $y = st$

<u>Thm</u> (full version)

Let $u = u(x_1, \ldots, x_n)$ where each $x_i = x_i(t_1, \ldots, t_m)$. Then u is also a fcn of $t_1, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

<u>Ex.</u> $P = u^2 + 3v^2 - 4w^2$

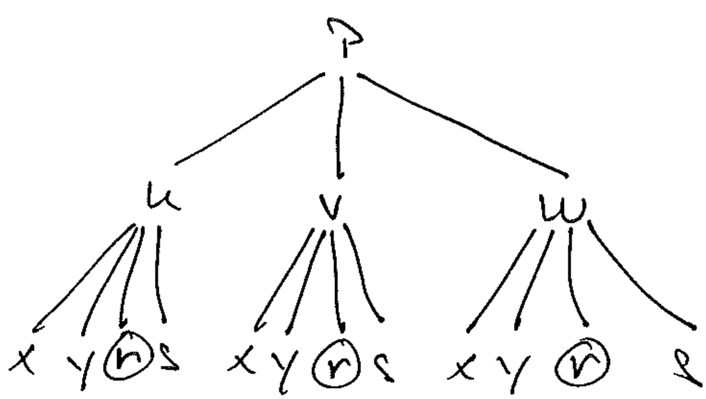
$$u = x - 3y + 2r - s$$
$$v = 2x + y - r - 2s$$
$$w = -x + 2y + r + s$$

<u>Find</u>

$\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}$

$\frac{\partial P}{\partial r}, \frac{\partial P}{\partial s}$

<u>DEPENDENCY TREE</u>



$$\frac{\partial P}{\partial r} = \frac{\partial P}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial P}{\partial v} \cdot \frac{\partial v}{\partial r} + \frac{\partial P}{\partial w} \cdot \frac{\partial w}{\partial r}$$
$$= 2u \cdot 2 + 6v \cdot (-1) - 8w \cdot 1$$
$$= 4(x - 3y + 2r - s) - 6(2x + y - r - 2s) - 8(-x + 2y + r + s)$$

$= 0\cdot x - 34y + 6r + 0\cdot s$
$= -34y + 6r$

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial P}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial P}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$= 2u\cdot 1 + 6v\cdot 2 - 8w\cdot(-1)$
$= 2(x-3y+2r-s) + 12(2x+y-r-2s) + 8(-x+2y+r+s)$
$= (2+24-8)x + (-6+12+16)y + (4-12+8)r + (-2-24+8)s$

$= 18x + 22y - 18s$

EX. LET $z = f(x,y)$ HAVE CONTINUOUS 2ND PARTIAL DERIVATIVES, AND SUPPOSE

$x = s^2 + t^2$, $y = 2st$

FIND $\frac{\partial^2 z}{\partial s^2}$. (NOTE: $f_{xy} = f_{yx}$).

$$\frac{\partial z}{\partial s} = f_x\cdot\frac{\partial x}{\partial s} + f_y\frac{\partial y}{\partial s}$$

$$\frac{\partial^2 z}{\partial s^2} = f_x\cdot\frac{\partial^2 x}{\partial s^2} + \left(f_{xx}\cdot\frac{\partial x}{\partial s} + f_{xy}\frac{\partial y}{\partial s}\right)\cdot\frac{\partial x}{\partial s} + f_y\cdot\frac{\partial^2 y}{\partial s^2} + \left(f_{yx}\frac{\partial x}{\partial s} + f_{yy}\frac{\partial y}{\partial s}\right)\cdot\frac{\partial y}{\partial s}$$

$$= f_{xx}\cdot\left(\frac{\partial x}{\partial s}\right)^2 + 2f_{xy}\cdot\frac{\partial x}{\partial s}\frac{\partial y}{\partial s} + f_{yy}\cdot\left(\frac{\partial y}{\partial s}\right)^2 + f_x\cdot\frac{\partial^2 x}{\partial s^2} + f_y\cdot\frac{\partial^2 y}{\partial s^2}$$

$$= \boxed{2\cdot f_x + 4s^2 f_{xx} + 8st f_{xy} + 4t^2 f_{yy}}$$

IMPLICIT DIFFERENTIATION

SUPPOSE THE EQN. $f(x,y)=0$ DEFINES y IMPLICITELY AS A FCN. OF x. i.e. IF WE COULD SOLVE

$$* \qquad f(x,y)=0$$

FOR y AS $y=g(x)$, THEN WE WOULD HAVE

$$** \qquad f(x,g(x))=0$$

FOR ALL $x \in \mathrm{Dom}(g)$. SUPPOSE ALSO THAT THE FCN. g IS DIFF.

DIFF. BOTH SIDES OF $*$ WRT x GIVES, BY THE CHAIN RULE;

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

i.e. $$\boxed{\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}}$$

THIS PROCESS IS KNOWN AS IMPLICIT DIFFERENTIATION

USING THE NOTATION IN $**$ WE HAVE

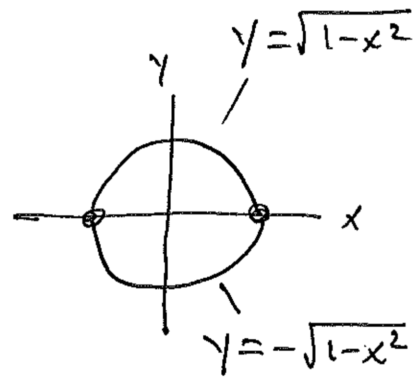
$$f_x \cdot 1 + f_y \cdot g'(x) = 0$$

$$\therefore \quad g'(x) = -\frac{f_x}{f_y}.$$

__Ex__ $f(x,y) = x^2 + y^2 - 1 = 0$

__Implicitly__: $f_x = 2x,\ f_y = 2y$

$$\therefore \quad \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$



__Explicitly__

$$y = (1-x^2)^{1/2} \qquad \text{or} \qquad y = -(1-x^2)^{1/2}$$

$$\frac{dy}{dx} = \tfrac{1}{2}(1-x^2)^{-1/2}(-2x) \qquad \frac{dy}{dx} = -\tfrac{1}{2}(1-x^2)^{-1/2}(2x)$$

$$= \frac{-x}{(1-x^2)^{1/2}} \qquad = \frac{-x}{-(1-x^2)^{1/2}}$$

$$= -\frac{x}{y} \qquad = -\frac{x}{y}$$

OFTEN THOUGH THE EXPLICIT APPROACH IS DIFFICULT, IF NOT IMPOSSIBLE.

EX. FIND SLOPE OF TAN. LINE TO CURVE

$$xy e^{xy} = \ln 4$$

AT THE POINT $(1, \ln 2)$.

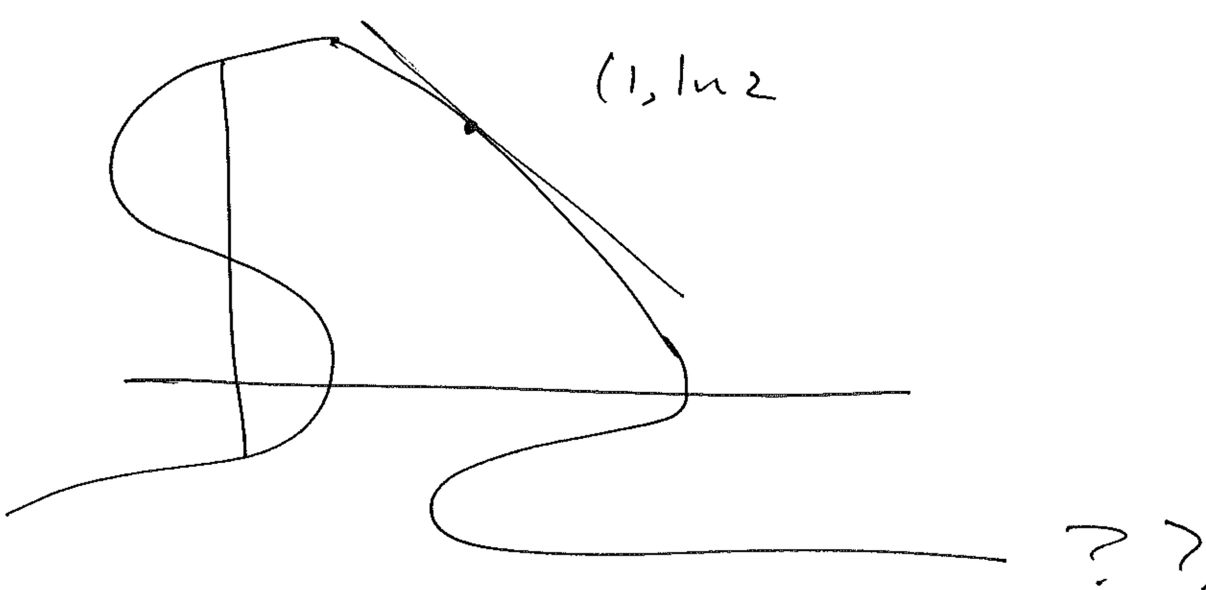
$$f(x,y) = xy e^{xy} - \ln 4 = 0$$

$$f_x = xy^2 e^{xy} + y e^{xy} = y(xy+1)e^{xy}$$

$$f_y = x^2 y e^{xy} + x e^{xy} = x(xy+1)e^{xy}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y(xy+1)e^{xy}}{x(xy+1)e^{xy}} = -\frac{y}{x}$$

$$\left.\frac{dy}{dx}\right|_{(1,\ln 2)} = -\ln 2$$



NOW SUPPOSE $F(x,y,z)=0$ DEFINES z IMPLICITLY AS A FCN OF BOTH x AND y (WHERE x, y VARY INDEPENDENTLY.) THEN WON'T DIFF. WRT x.

$$F_x \cdot \frac{\partial x}{\partial x} + F_y \cdot \frac{\partial y}{\partial x} + F_z \cdot \frac{\partial z}{\partial x} = 0$$

BUT $\frac{\partial x}{\partial x} = 1$ & $\frac{\partial y}{\partial x} = 0$, SO

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

similarly

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

Ex. $3xy^2 + 2yz^2 + 5x^2z - xyz = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{3y^2 + 10xz - yz}{4yz + 5x^2 - xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{6xy + 2z^2 - xz}{4yz + 5x^2 - xy}$$