(14.2) LIMITS AND CONTINUITY

IN THIS SECTION WE WILL DEFINE WHAT IS MEANT BY THE STATEMENT

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

PROBLEMS ARISE WHEN THE LIMIT L DEPENDS ON THE WAY THE POINT (x, y) 'APPROACHES' THE POINT (a, b). WHEN THIS OCCURS, WE WILL SAY THAT THE ABOVE LIMIT 'DOES NOT EXIST.'

EX. CONSIDER $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)$

ALONG THE LINE $x=0$ WE HAVE

$$\frac{x^2-y^2}{x^2+y^2} = \frac{-y^2}{y^2} = -1 \longrightarrow -1$$

while ALONG $y=0$ WE HAVE

$$\frac{x^2-y^2}{x^2+y^2} = \frac{x^2}{x^2} = +1 \longrightarrow +1$$

TAKING THE LIMIT ALONG ANY (NON-VERTICAL) LINE $y = mx$ GIVES

$$\frac{x^2-y^2}{x^2+y^2} = \frac{x^2-m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m^2} \to \frac{1-m^2}{1+m^2}$$

SINCE THIS LIMIT CLEARLY DEPENDS ON THE SLOPE m, WE MUST CONCLUDE THAT

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)$$

DOES NOT EXIST.

<u>DEFN</u>
LET f BE A FUNCTION OF 2 VARIABLES SUCH THAT $\mathrm{Dom}(f)$ CONTAINS POINTS ARBITRARILY CLOSE TO $(a,b) \in \mathbb{R}^2$. WE WRITE

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

OR

$$f(x,y) \to L \quad \text{AS} \quad (x,y) \to (a,b)$$

IFF FOR ALL $\varepsilon > 0$, THERE EXISTS $\delta > 0$ SUCH THAT:

IF $(x,y) \in D$ AND $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$,

THEN $|f(x,y) - L| < \varepsilon$.

RECALL THAT THE DISTANCE FROM (x,y) TO (a,b) IS $\sqrt{(x-a)^2+(y-b)^2}$.

THUS THE DEFINITION SAYS THAT $f(x,y)$ CAN BE MADE ARBITRARILY CLOSE TO L (i.e. for all $\varepsilon > 0$) BY TAKING (x,y) SUFFICIENTLY CLOSE TO (a,b) (i.e. THERE EXISTS $\delta > 0$).

OBSERVE THAT THIS DEFINITION MAKES NO MENTION OF THE DIRECTION OR MANNER OF APPROACH. THUS THE LIMIT L MUST BE THE SAME FOR ALL APPROACHING PATHS.

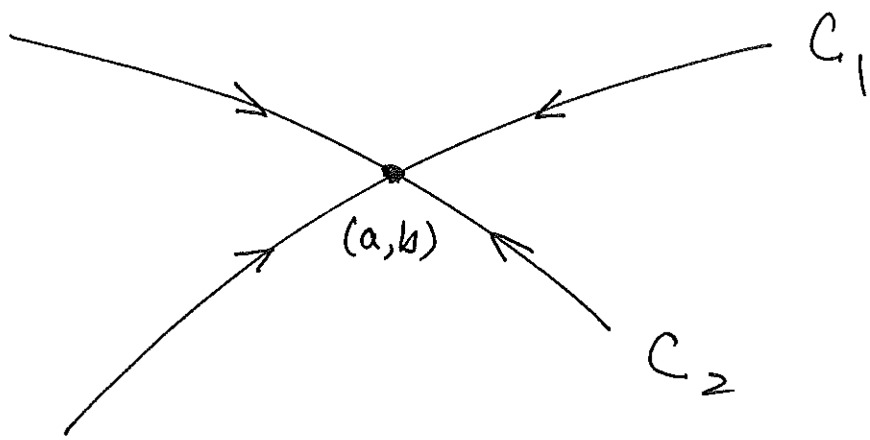
IN PARTICULAR: IF C_1, C_2 ARE CURVES IN $\mathbb{R}^2$ WHICH CONTAIN THE POINT (a,b) AND IF

$f(x,y) \to L_1$ AS $(x,y) \to (a,b)$ ALONG C_1,
AND
$f(x,y) \to L_2$ AS $(x,y) \to (a,b)$ ALONG C_2,

AND IF $L_1 \neq L_2$, THEN

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

DOES NOT EXIST.



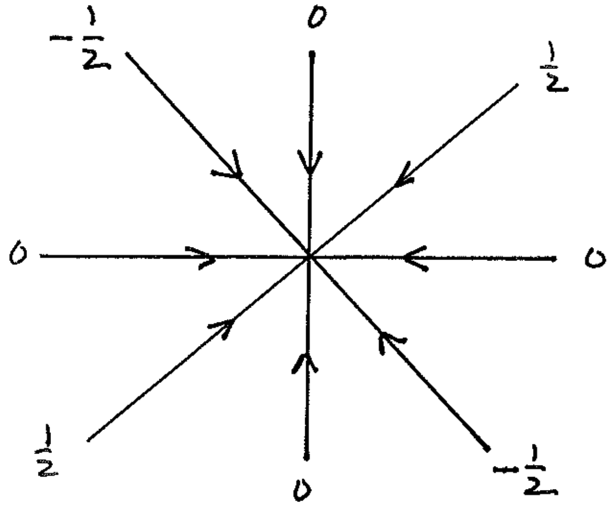
<u>Ex.</u> $f(x,y) = \dfrac{xy}{x^2+y^2}$ NOTE $(0,0) \notin Dom(f)$.

$y = mx \Rightarrow f(x,mx) = \dfrac{m^2x^2}{x^2+m^2x^2} = \dfrac{m}{1+m^2}$

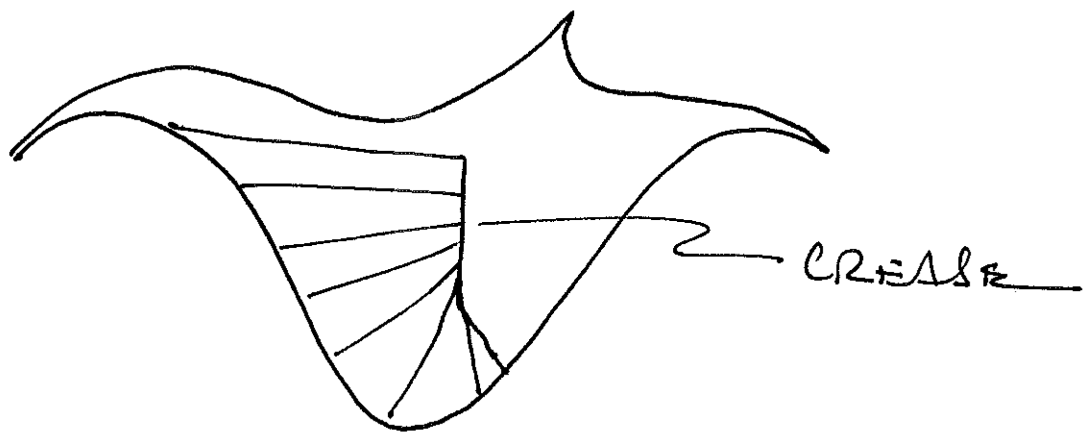
SINCE LIMIT DEPENDS ON THE SLOPE m,

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

DOES NOT EXIST.



$m = \pm 1$
$\Rightarrow \lim = \pm \frac{1}{2}$



Ex. $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ $\quad (0,0) \notin \mathrm{Dom}(f)$

ALONG THE LINE $y = mx$ WE HAVE

$$f(x,mx) = \frac{m^2x^3}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^2} \rightarrow 0 \text{ AS } x \rightarrow 0$$

THIS PROVES NOTHING! ALONG THE CURVE $x = y^2$ WE HAVE

$$f(y^2,y) = \frac{y^4}{y^4+y^4} = \frac{y^4}{2y^4} = \frac{1}{2} \rightarrow \frac{1}{2} \text{ AS } y \rightarrow 0$$

THUS $\lim\limits_{(x,y)\rightarrow(0,0)} f(x,y)$ D.N.E.

EX 4 P.873 SHOWS THAT A limit EXISTS USING THE DEFINITION

i.e. $\lim_{(x,y)\to(0,0)} \left(\frac{3x^2y}{x^2+y^2}\right) = 0$

THE FOLLOWING FACTS ARE CONSEQUENCES OF THE DEFINITION

- $\lim_{(x,y)\to(a,b)} const = const$
- $\lim_{(x,y)\to(a,b)} x = a$
- $\lim_{(x,y)\to(a,b)} y = b$
- SQUEEZE THEOREM:

IF $L \le f(x,y) \le g(x,y)$, AND IF

$$\lim_{(x,y)\to(a,b)} g(x,y) = L$$

THEN $\lim_{(x,y)\to(a,b)} f(x,y) = L$

WE ALSO HAVE THE FAMILIAR RESULTS FROM THE THEORY OF 1-DIMENSIONAL LIMITS

- THE LIMIT OF A SUM EQUALS THE SUM OF THE LIMITS.
- LIKEWISE FOR DIFFERENCE, PRODUCT, AND QUOTIENT OF LIMITS.

Ex. $$\lim_{(x,y)\to(1,3)} \left(\frac{xy^2}{x^2+y^4}\right) = \frac{1\cdot 3^2}{1^2+3^4} = \frac{9}{82}$$

Ex. $$\lim_{(x,y)\to(0,0)} \left(\frac{x^4}{x^2+y^2}\right) = 0$$

USING THE SQUEEZE THEOREM

$$0 \le \frac{x^4}{x^2+y^2} \le \frac{x^4}{x^2} = x^2 \to 0 \text{ AS } x \to 0$$

$\uparrow$

SINCE $y^2 \ge 0$.

EX. $$\lim_{(x,y)\to(0,0)} \left(\frac{3x^2y}{x^2+y^2}\right) = 0$$

ALSO FOLLOWS FROM THE SQUEEZE THEOREM.

$$0 \le \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \le \frac{3(x^2+y^2)|y|}{(x^2+y^2)} = 3|y|$$

AND $3|y| \to 0$ AS $(x,y) \to (0,0)$.

DEFN.
A FUNCTION $f : D \to \mathbb{R}$ IS SAID TO BE CONTINUOUS AT $(a,b) \in D$ IFF

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

WE SAY $f(x,y)$ IS CONTINUOUS ON D IFF f IS CONTINUOUS AT EACH POINT $(a,b) \in D$.

THM: A POLYNOMIAL $P(x,y)$ IN 2 VARIABLES IS CONTINUOUS ON $\mathbb{R}^2$.

RECALL THAT A <u>RATIONAL FUNCTION</u> IS A RATIO OF TWO POLYNOMIALS

$$f(x,y) = \frac{P(x,y)}{Q(x,y)}$$

<u>THM</u>

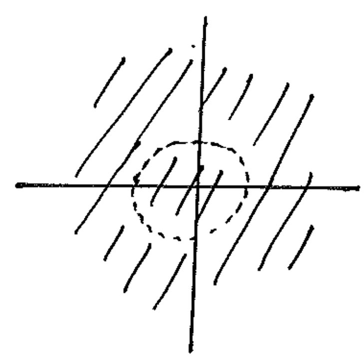
A RATIONAL FUNCTION IN 2 VARIABLES IS CONTINUOUS ON ITS DOMAIN, i.e. AT ALL POINTS $(a,b) \in \mathbb{R}^2$ SUCH THAT $Q(a,b) \neq 0$.

<u>THM</u>

THE COMPOSITION OF CONTINUOUS FUNCTIONS IS CONTINUOUS

<u>Ex.</u> $f(x,y) = \cos\left(\frac{x^2+y^4}{1-x^2-y^2}\right)$

$\mathrm{Dom}(f) = \{(x,y) \mid 1-x^2-y^2 \neq 0\}$



e.g. $\lim_{(x,y)\to(0,0)} f(x,y) = \cos\left(\frac{0}{1}\right) = 1$

WE HAVE SIMILAR DEFINITIONS OF LIMITS AND CONTINUITY FOR FUNCTIONS OF $3, 4, \ldots, n$ VARIABLES.