(14.1) FUNCTIONS OF SEVERAL VARIABLES

RECALL THE GENERAL DEFINITION OF A FUNCTION

DEFN

A FUNCTION f CONSISTS OF:

(1) A SET D CALLED THE DOMAIN

(2) A SET C CALLED THE CO-DOMAIN

(3) A RULE f WHICH ASSIGNS TO EACH $\alpha \in D$ A UNIQUE $\beta \in C$ CALLED THE IMAGE OF α UNDER f. WE OFTEN WRITE $\beta = f(\alpha)$.

WE WRITE $f: D \to C$ TO MEAN THAT f IS A FUNCTION WITH DOMAIN D AND CO-DOMAIN C. WE ALSO WRITE

$$\mathrm{Dom}(f) = D$$
$$\mathrm{Codom}(f) = C$$

DEFN

THE RANGE OF f IS THE SET OF ALL IMAGES:

$$\mathrm{Range}(f) = \{ f(\alpha) \mid \alpha \in \mathrm{Dom}(f) \}.$$

OBSERVE THAT $\mathrm{Range}(f) \subseteq \mathrm{Codom}(f)$.

A FUNCTION f IS SAID TO <u>MAP</u> THE POINT $\alpha \in \mathrm{Dom}(f)$ TO THE POINT $f(\alpha) \in \mathrm{Codom}(f)$. FOR THIS REASON FUNCTIONS ARE ALSO CALLED <u>MAPS</u> OR <u>MAPPINGS</u>.

WE SAY f IS A FUNCTION OF 2 VARIABLES IF

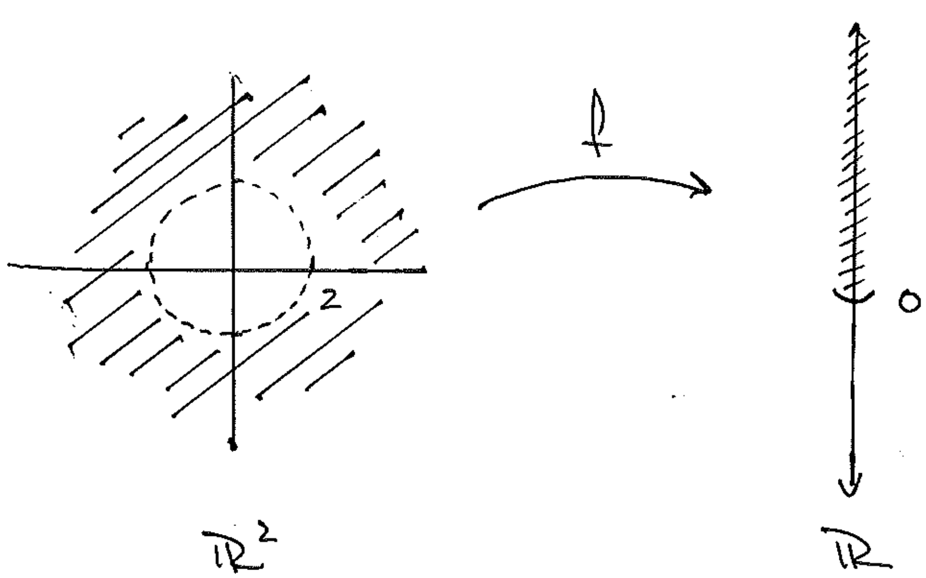
$$\mathrm{Dom}(f) \subseteq \mathbb{R}^2$$

<u>EX.</u>

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2-4}}$$

$$\mathrm{Dom}(f) = \{(x,y) \mid x^2+y^2-4 > 0\} \subseteq \mathbb{R}^2$$

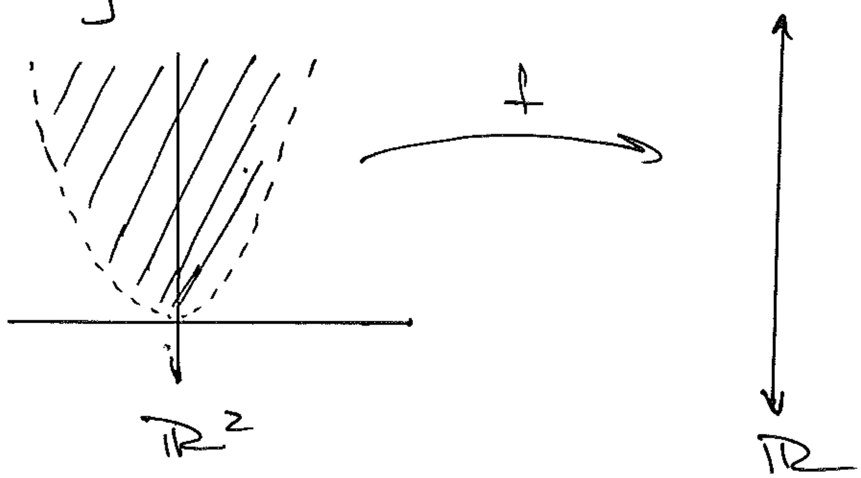
$$\mathrm{Range}(f) = (0,\infty) \subseteq \mathbb{R}$$



Ex. $g(x,y) = \ln(y - x^2)$

$\mathrm{Dom}(g) = \{(x,y) \mid y - x^2 > 0\}$

$\mathrm{Range}(g) = \mathbb{R}$



DEFN
THE GRAPH OF A FUNCTION f IS THE SET

$$\mathrm{Graph}(f) = \{(\alpha, f(\alpha)) \mid \alpha \in \mathrm{Dom}(f)\}$$

IF f IS A FUNCTION OF 2 VARIABLES THEN

$$\mathrm{GRAPH}(f) = \{((x,y), f(x,y)) \mid (x,y) \in \mathrm{Dom}(f)\}$$

$$= \{(x,y,z) \mid z = f(x,y)\}$$

WHICH IS A SURFACE IN $\mathbb{R}^3$.

Ex. $f(x,y) = 5 - \frac{3}{2}x - \frac{5}{2}y$

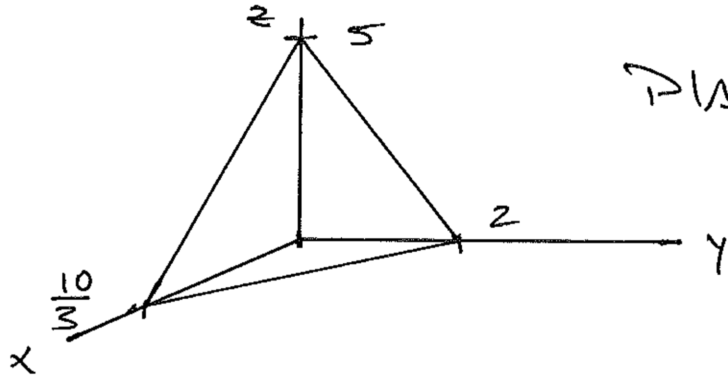
$\text{Dom}(f) = \mathbb{R}^2$, $\text{Range}(f) = \mathbb{R}$

Graph(f) is THE SURFACE WITH EQUATION

$$z = 5 - \frac{3}{2}x - \frac{5}{2}y$$

i.e.

$$3x - 5y + 2z = 10$$

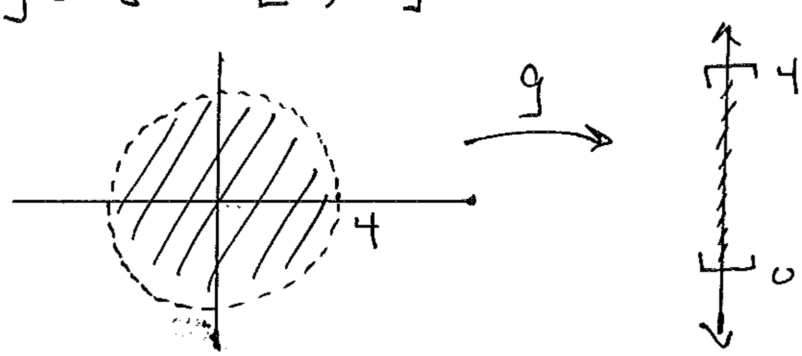


PLANE

Ex. $g(x,y) = \sqrt{16 - x^2 - y^2}$

$\text{Dom}(g) = \{(x,y) \mid 16 - x^2 - y^2 \geq 0\}$

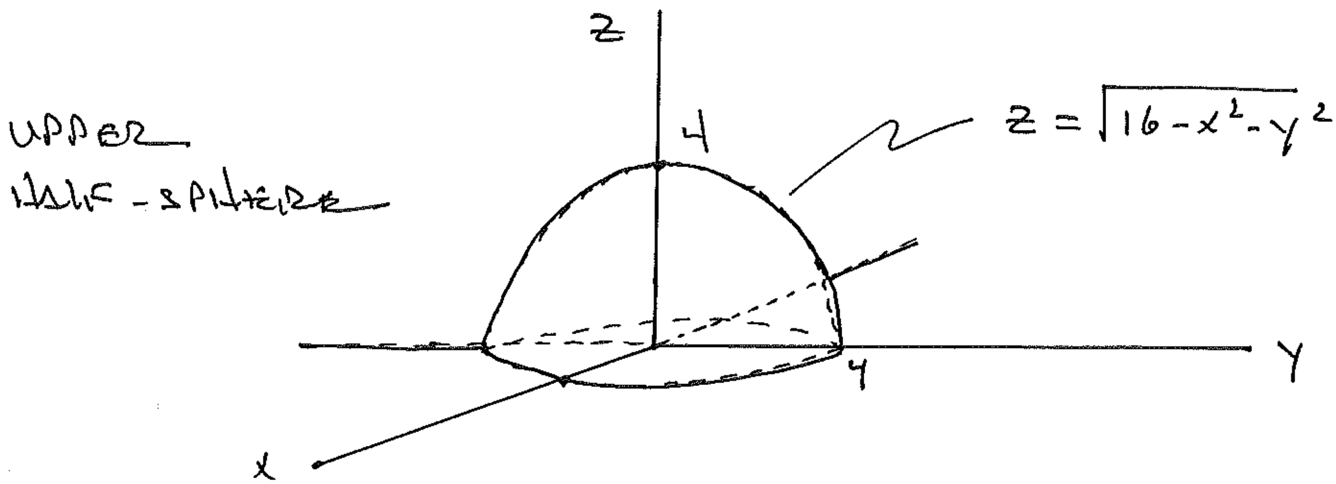
$\text{Range}(g) = [0, 4]$



Graph(g) is the surface

$$z = \sqrt{16 - x^2 - y^2}$$

i.e. $x^2+y^2+z^2=4^2$, $z \geq 0$

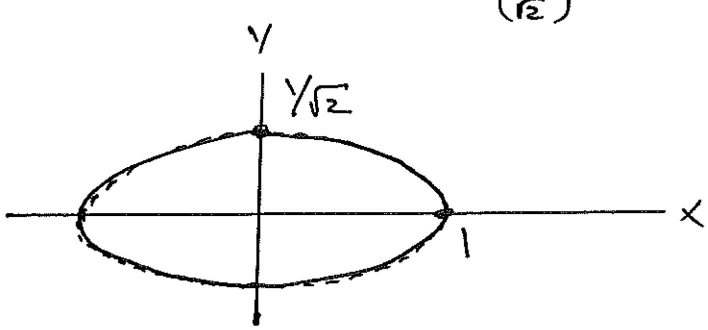


Ex. $h(x,y) = x^2+2y^2$

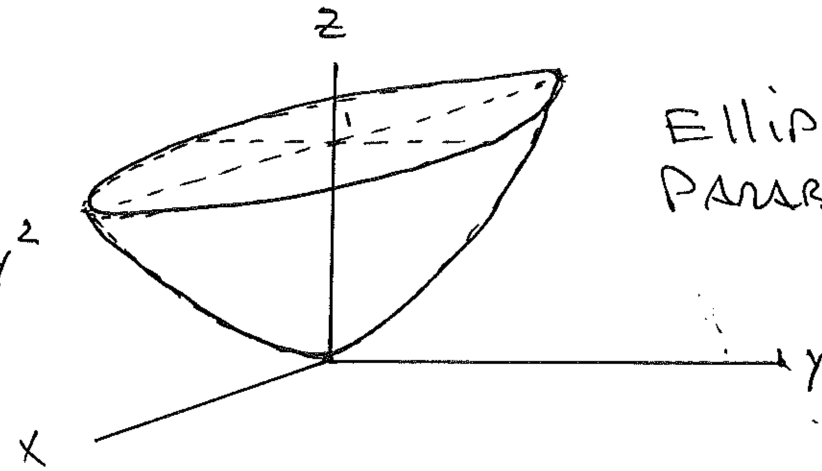
$\text{Dom}(h) = \mathbb{R}^2$, $\text{Range}(h) = [0,\infty)$.

Graph(h) : $z = x^2+2y^2$

Trace $z=1$: $x^2 + \dfrac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1$



$z = x^2 + 2y^2$



Elliptic Paraboloid

<u>Defn</u>

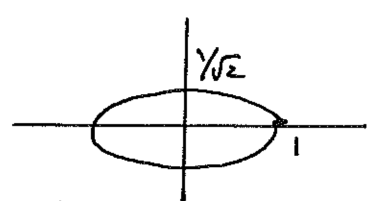
The level curves of a function of two variables $f(x,y)$ are the curves in $\mathbb{R}^2$ with equations

$$f(x,y) = k$$

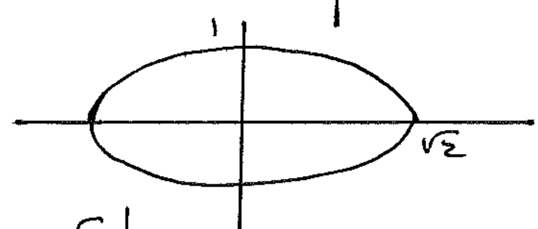
where k is a constant, $k \in \text{Range}(f)$.

<u>Ex.</u> $h(x,y) = x^2 + 2y^2$

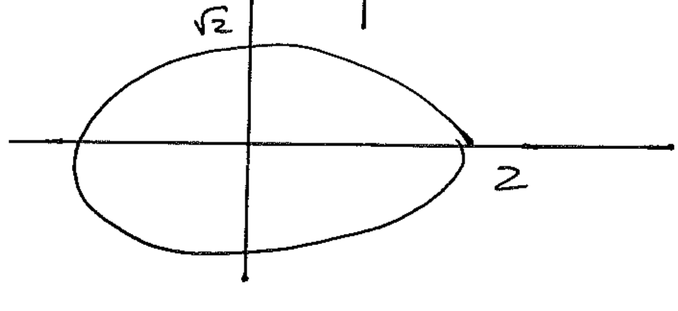
$k = 1 : \ x^2 + 2y^2 = 1$



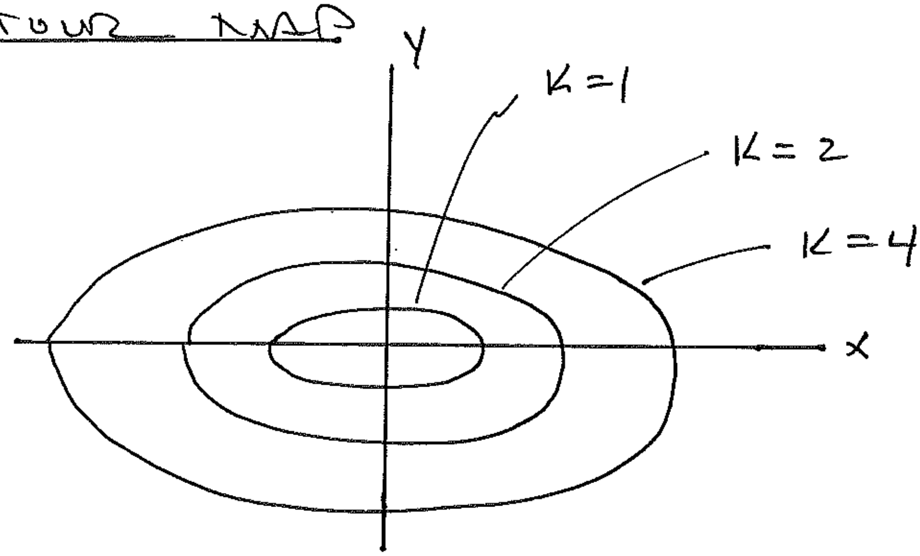
$k = 2 : \ x^2 + 2y^2 = 2$



$k = 4 : \ x^2 + 2y^2 = 4$



CONTOUR MAP



WE SAY f IS A FUNCTION OF 3 VARIABLES IFF

$$\mathrm{Dom}(f) \subseteq \mathbb{R}^3$$

AND

$$\mathrm{Range}(f) \subseteq \mathbb{R}$$

IN THIS CASE $\mathrm{Graph}(f) \subseteq \mathbb{R}^4$, AND CAN BE DIFFICULT TO VISUALIZE.

WE SPEAK OF THE <u>LEVEL SURFACES</u> OF SUCH A FUNCTION, WHICH ARE SOLUTIONS TO EQUATIONS

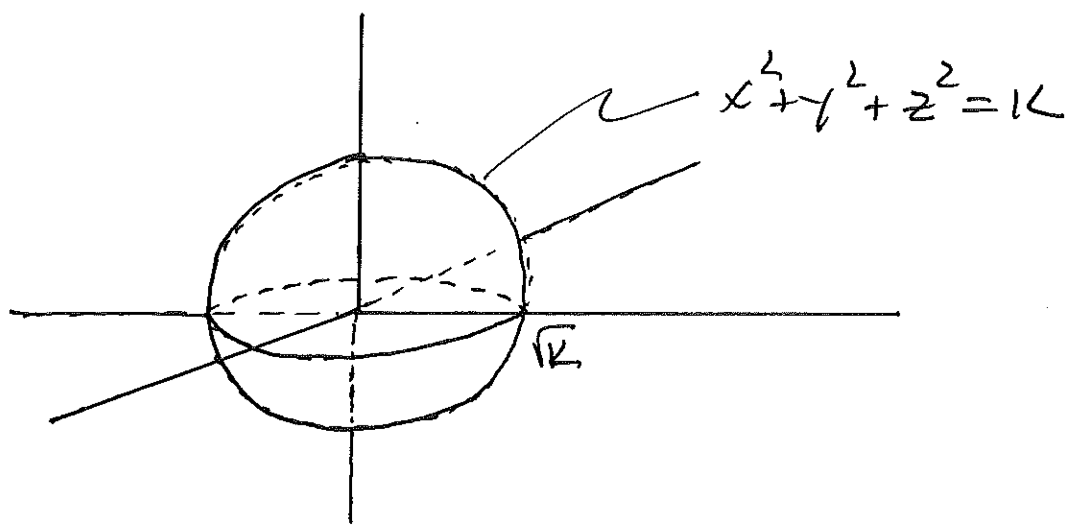
$$f(x, y, z) = K$$

WITH $K \in \mathrm{Range}(f)$. THESE ARE SURFACES IN $\mathbb{R}^3$.

EX. $f(x,y,z) = x^2+y^2+z^2$

$\mathrm{Dom}(f) = \mathbb{R}^3$, $\mathrm{Range}(f) = [0,\infty)$

THE LEVEL SURFACE $f(x,y,z) = K$ $(K \geq 0)$ IS A SPHERE ABOUT $(0,0,0)$ OF RADIUS $\sqrt{K}$.



MORE GENERALLY WE SAY f IS A FUNCTION OF n VARIABLES IF $\mathrm{Dom}(f) \subseteq \mathbb{R}^n$ AND $\mathrm{Range}(f) \subseteq \mathbb{R}$.

NOTE: $\mathbb{R}^n = \{ \underbrace{(x_1, x_2, \ldots, x_n)} \mid x_i \in \mathbb{R},\ 1 \leq i \leq n \}$

AN n-TUPLE OF REAL NUMBERS

In This CASE $\mathrm{Graph}(f) \subseteq \mathbb{R}^{n+1}$,

AND THE 'LEVEL SETS' OF f ARE $(n-1)$-DIMENSIONAL 'HYPER-SURFACES' LYING IN $\mathbb{R}^n$.

Ex

$$f(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2$$

THE level SETS $f(x_1, x_2, \dots, x_n) = K$ ARE $(n-1)$-DIMENSIONAL SPHERES LYING IN $\mathbb{R}^n$.