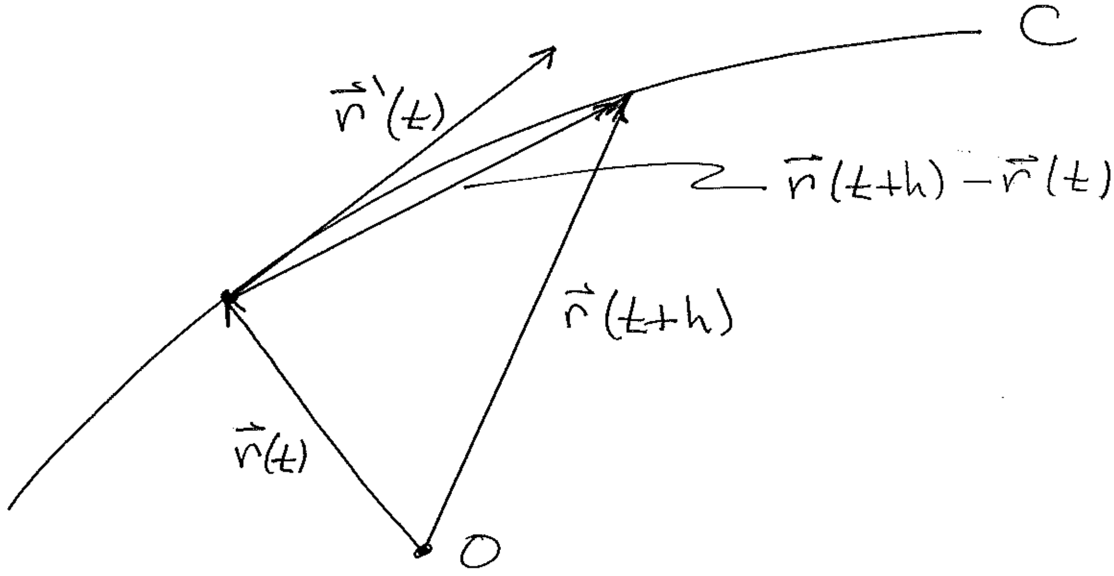
(13.2) DERIVATIVES & INTEGRALS

LET $\vec{r}(t)$ BE A VECTOR FUNCTION PARAMETRIZING A CURVE C IN $\mathbb{R}^3$

DEFN

$$\frac{d}{dt}[\vec{r}(t)] = \vec{r}\,'(t) = \lim_{h\to 0}\left(\frac{\vec{r}(t+h)-\vec{r}(t)}{h}\right)$$



WE SEE THAT $\vec{r}\,'(t)$ IS TANGENT TO C AT $\vec{r}(t)$.

AS WE'LL SEE $|\vec{r}\,'(t)|$ IS THE INSTANTANEOUS SPEED WITH WHICH $\vec{r}(t)$ TRACES THE CURVE C.

THM

IF $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ THEN

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

PROOF:

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{1}{h}\left(\vec{r}(t+h) - \vec{r}(t) \right)$$

$$= \lim_{h \to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \quad \cdot \quad, \quad \cdot \quad \right\rangle$$

$$= \left\langle \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \quad \cdot \quad, \quad \cdot \quad \right\rangle$$

$$= \langle f'(t), \quad \cdot \quad, \quad \cdot \quad \rangle$$

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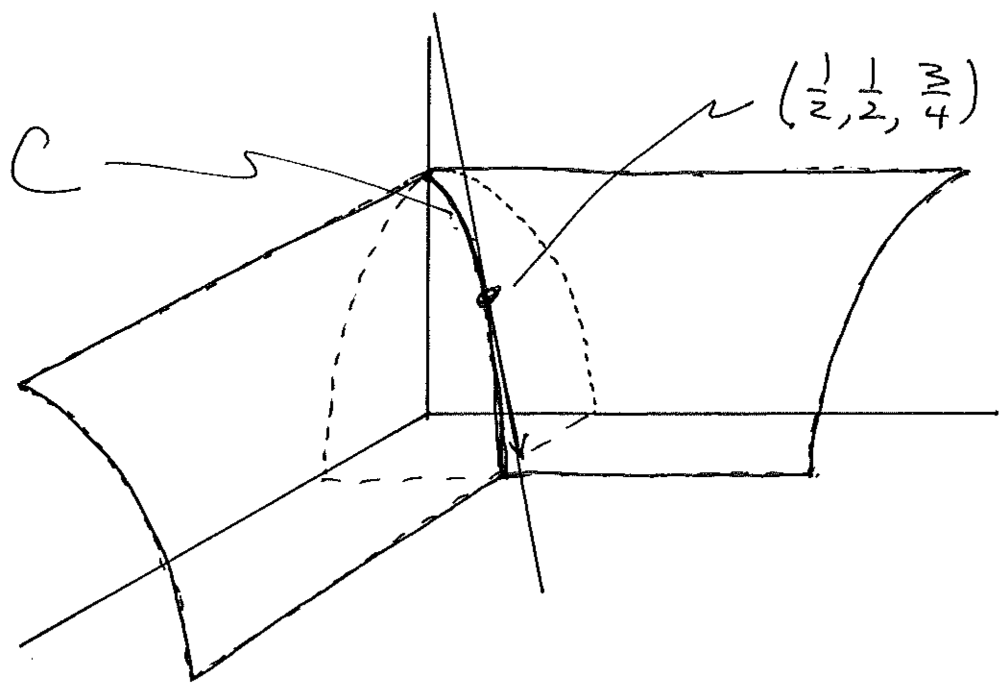
DEFN

WE CALL

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

THE UNIT TANGENT VECTOR TO C AT $\vec{r}(t)$.

Ex.
LET C BE THE PART OF THE INTERSECTION OF THE PARABOLIC CYLINDERS $z = 1 - x^2$ AND $z = 1 - y^2$ WHICH LIES IN THE FIRST OCTANT.



FIND THE UNIT TANGENT VECTOR TO C AT $(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})$, AS WELL AS THE TANGENT LINE AT THE SAME POINT.

PARAMETERIZE C BY

$$\vec{r}(t) = \langle t, t, 1 - t^2 \rangle$$

FOR $0 \leq t \leq 1$.

THEN $\vec{r}\left(\frac{1}{2}\right) = \left\langle \frac{1}{2}, \frac{1}{2}, \frac{3}{4} \right\rangle$,

$$\vec{r}\,'(t) = \langle 1, 1, -2t \rangle$$

AND

$$\vec{r}\,'\left(\tfrac{1}{2}\right) = \langle 1, 1, -1 \rangle .$$

THE UNIT TANGENT VECTOR IS

$$\vec{T}\left(\frac{1}{2}\right) = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

AND THE TANGENT LINE HAS SYMMETRIC EQUATION

$$x - \frac{1}{2} = y - \frac{1}{2} = \frac{3}{4} - z$$

THE SECOND DERIVATIVE OF A VECTOR FUNCTION IS

$$\vec{r}\,''(t) = \frac{d}{dt}\left[\vec{r}\,'(t)\right]$$

$$= \langle f''(t), g''(t), h''(t) \rangle$$

THM

LET $\vec{u}(t)$, $\vec{v}(t)$ BE DIFFERENTIABLE VECTOR FUNCTIONS. THEN

1.) $\frac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \vec{u}'(t)+\vec{v}'(t)$

2.) $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\vec{u}'(t)$

3.) $\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

4.) $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

5.) $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

6.) $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}'(f(t))$

All PROOFS FOLLOW BY APPLYING THE CORRESPONDING FACT TO EACH COMPONENT.

e.g. PROOF OF #6!

$$\frac{d}{dt}\left[\vec{u}(f(t))\right]$$

$$= \frac{d}{dt}\left\langle u_1(f(t)),\, u_2(f(t)),\, u_3(f(t))\right\rangle$$

$$= \left\langle \frac{d}{dt} u_1(f(t)) \,,\ \cdot \,,\ \cdot \right\rangle$$

$$= \left\langle f'(t)\, u_1'(f(t)) \,,\ \cdot \,,\ \cdot \right\rangle$$

$$= f'(t) \left\langle u_1'(f(t)) \,,\ \cdot \,,\ \cdot \right\rangle$$

$$= f'(t)\, \vec{u}\,'(t).$$

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<u>THM</u>

IF $\vec{r}(t)$ LIES ON A SPHERE WITH CENTER $(0,0,0)$ THEN

$$\vec{r}(t) \perp \vec{r}\,'(t)$$

FOR ALL t.

<u>PROOF:</u>

WE HAVE $|\vec{r}(t)| = R$ FOR ALL t, WHENCE

$$0 = \frac{d}{dt}\left[R^2\right] = \frac{d}{dt} |\vec{r}(t)|^2$$

$$= \frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right)$$

$$= \vec{r}(t)\cdot\vec{r}'(t) + \vec{r}'(t)\cdot\vec{r}(t)$$

$$= 2\,\vec{r}(t)\cdot\vec{r}'(t)$$

$\therefore\ \vec{r}(t)\cdot\vec{r}'(t) = 0$ FOR ALL t. ///

DEFN:
THE INTEGRAL OF A VECTOR FUNCTION IS DEFINED COMPONENT-WISE

GIVEN $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, THEN

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

EX. $\vec{r}(t) = \langle t, t, 1-t^2\rangle \quad 0 \le t \le 1$

$$\int_0^1 \vec{r}(t)\,dt = \left\langle \frac{t^2}{2}\Big|_0^1, \frac{t^2}{2}\Big|_0^1, \left(t-\frac{t^3}{3}\right)\Big|_0^1 \right\rangle$$

$$= \left\langle \frac{1}{2}, \frac{1}{2}, \frac{2}{3} \right\rangle.$$