## 12.5 LINES AND PLANES

LET $L$ BE THE LINE IN $\mathbb{R}^3$ THROUGH THE POINT $P(x_0, y_0, z_0)$ AND PARALLEL TO $\vec{v} = \langle a, b, c \rangle$. LET $Q(x, y, z)$ BE ANY POINT ON $L$, AND

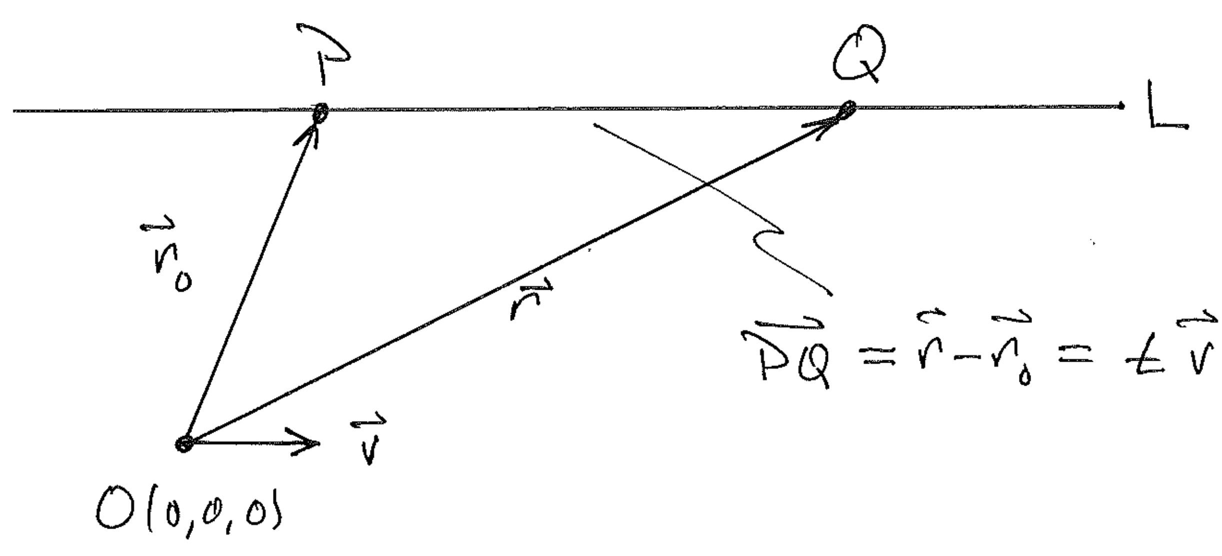
$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle \quad , \quad \vec{r} = \langle x, y, z \rangle$$

BE THE POSITION VECTORS OF $P$ AND $Q$ RESPECTIVELY.

SINCE $\overrightarrow{PQ} = \vec{r} - \vec{r}_0$ IS ALSO IN THE DIRECTION OF $L$, IT MUST BE A SCALAR MULTIPLE OF $\vec{v}$, i.e.

$$\vec{r} - \vec{r}_0 = t\,\vec{v}$$

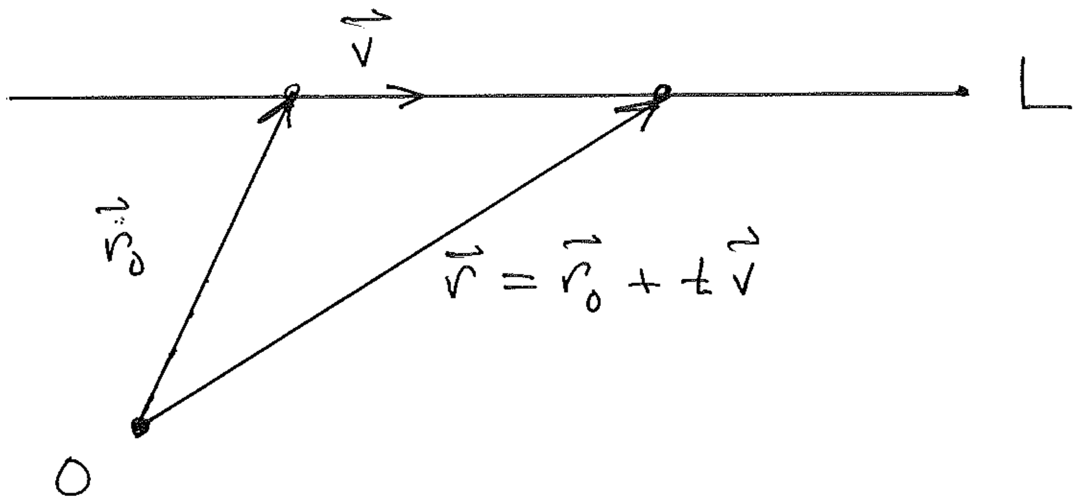
FOR SOME $t \in \mathbb{R}$.



THUS

(1) $\vec{r} = \vec{r}_0 + t\vec{v}$ $\left\{ \begin{array}{l} \text{VECTOR EQN.} \\ \text{FOR } L \end{array} \right.$

AS $t \in \mathbb{R}$ RANGES OVER ALL REALS, WE SEE THAT $\vec{r}$ "DRAWS" THE LINE $L$




IF WE WRITE THIS VECTOR EQUATION IN COMPONENT FORM WE GET THE PARAMETRIC EQUATIONS FOR L

(2) $$\left\{ \begin{array}{l} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{array} \right.$$

DEFN
WE CALL $\vec{v} = \langle a, b, c \rangle$ A DIRECTION VECTOR FOR L, AND $a, b, c$ DIRECTION NUMBERS FOR L.

IF WE ELIMINATE $t$ FROM EQUATIONS (2) WE GET THE SYMMETRIC EQUATIONS FOR L

(3) $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EX. WRITE PARAMETRIC AND SYMMETRIC EQUATIONS FOR THE LINE THROUGH $(1, -1, 5)$ PARALLEL TO $\langle 3, 2, -1 \rangle$

$$\begin{cases} x = 1 + 3t \\ y = -1 + 2t \\ z = 5 - t \end{cases}$$

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-5}{-1}$$

Ex. WRITE PARAMETRIC EQUATIONS FOR THE LINE THROUGH $(1,-1,5)$ THAT IS PERPENDICULAR TO BOTH $\langle 1,0,1\rangle$ AND $\langle 2,1,-3\rangle$

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 1 & -3 \end{vmatrix} = \langle -1, 5, 1\rangle$$

$$\begin{cases} x = 1 - t \\ y = -1 + 5t \\ z = 5 + t \end{cases}$$

Ex. DETERMINE THE INTERSECTIONS OF THE ABOVE LINE WITH THE COORDINATE PLANES.

$$\frac{x-1}{-1} = \frac{y+1}{5} = \frac{z-5}{1}$$

$xy$-PLANE : $z=0 \Rightarrow 1-x = \frac{y+1}{5} = -5$

$\Rightarrow x=6,\ y=-26$

$\Rightarrow (6,-26,0)$

Similarly

$xz$-PLANE : $(4/5, 0, 26/5)$

$yz$-PLANE : $(0, 4, 6)$ .

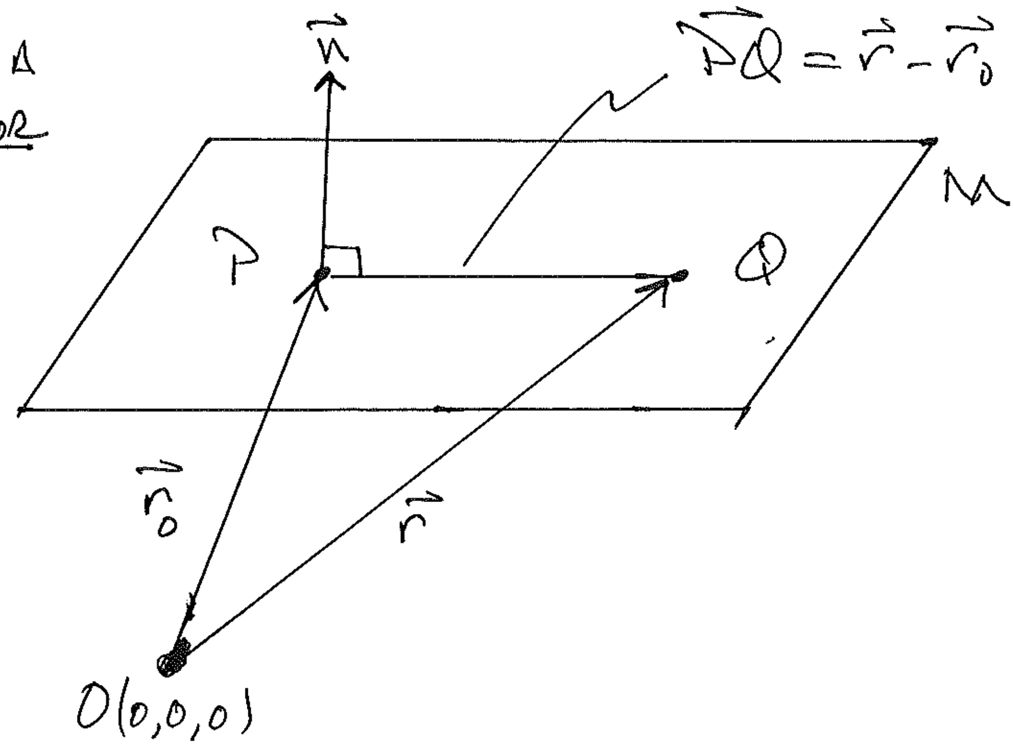
LET M BE THE PLANE IN $\mathbb{R}^3$ THROUGH $P(x_0, y_0, z_0)$ AND PERPENDICULAR TO $\vec{n} = \langle a, b, c \rangle$. LET $Q(x, y, z)$ BE ANY POINT ON M, AND AS BEFORE LET

$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle \quad , \quad \vec{r} = \langle x, y, z \rangle$$

BE THE POSITION VECTORS OF P AND Q. SINCE $\overrightarrow{PQ} = \vec{r} - \vec{r}_0$ IS PARALLEL TO M IT MUST BE PERPENDICULAR TO $\vec{n}$. THUS

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$\vec{n}$ IS CALLED A <u>NORMAL VECTOR</u> TO M



WRITING THIS EQUATION IN TERMS OF THE COMPONENTS OF $\vec{n}$ AND $\vec{r} - \vec{r}_0$ WE HAVE

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

OR EQUIVALENTLY

$$ax + by + cz = d$$

WHERE $d = ax_0 + by_0 + cz_0$ .

<u>EX.</u>

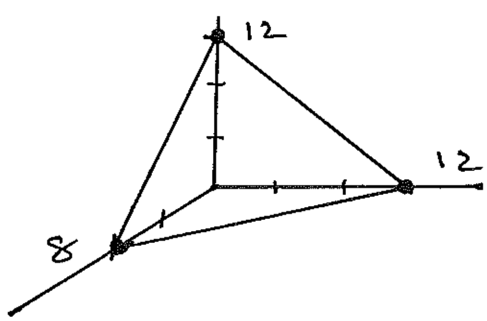
$P(2, -1, 10)$ , $\vec{n} = \langle 3, 2, 2 \rangle$

$$3(x - 2) + 2(y + 1) + 2(z - 10) = 0$$

i.e.

$$3x + 2y + 2z = 24$$

z-AXIS : $x = y = 0 \Rightarrow z = 12$
y-AXIS : $x = z = 0 \Rightarrow y = 12$
x-AXIS : $y = z = 0 \Rightarrow x = 8$



THE ANGLE BETWEEN TWO PLANES IS THE ANGLE BETWEEN THEIR NORMAL VECTORS.

EX. FIND THE ANGLE BETWEEN THE PLANES

$$x + y + 2z = 5$$
$$2x - y + z = 4$$

THE NORMAL VECTORS ARE
$\vec{u} = \langle 1, 1, 2 \rangle$ AND $\vec{v} = \langle 2, -1, 1 \rangle$
SO

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{2-1+2}{\sqrt{6} \cdot \sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$\therefore \quad \theta = \pi/3 = 60^\circ .$$

EX. FIND SYMMETRIC EQNS FOR THE LINE OF INTERSECTION OF THE SAME TWO PLANES.

THIS LINE HAS DIRECTION

$$\vec{u} \times \vec{v} = \langle 3, 3, -3 \rangle = 3 \langle 1, 1, -1 \rangle$$

AND PASSES THROUGH THE POINT $(1, 0, 2)$.

(TO SEE THIS ELIMINATE $y$ FROM

$$\begin{cases} x + y + 2z = 5 \\ 2x - y + z = 4 \end{cases}$$

TO GET $x + z = 3$. PICK $x = 1$, $z = 2$ AND SOLVE EITHER EQN. FOR $y$ TO GET $y = 0$. )

THE SYMMETRIC EQNS ARE

$$\frac{x-1}{1} = \frac{y-0}{1} = \frac{z-2}{-1}$$

i.e.

$$x - 1 = y = 2 - z .$$

<u>READ</u>: EX #8 P. 800 : DISTANCE FROM A POINT TO A PLANE.