(12.4) THE CROSS PRODUCT

DEFN

LET $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ BE VECTORS IN $\mathbb{R}^3$. THEIR CROSS PRODUCT IS DEFINED TO BE :

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

OBSERVE THAT $\vec{a} \times \vec{b}$ IS ITSELF A VECTOR IN $\mathbb{R}^3$. FOR THIS REASON IT IS ALSO SOMETIMES KNOWN AS THE VECTOR PRODUCT.

THE CROSS PRODUCT CAN ALSO BE DEFINED USING DETERMINANTS.

2×2 DETERMINANT :

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

e.g.

$$\begin{vmatrix} 1 & 5 \\ -2 & 4 \end{vmatrix} = 1 \cdot 4 - 5 \cdot (-2) = 14$$

3×3 DETERMINANT:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

e.g.

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 5 \\ -3 & -2 & 4 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 5 \\ -2 & 4 \end{vmatrix} - 0 \cdot \begin{vmatrix} & \\ & \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 \\ -3 & -2 \end{vmatrix}$$

$$= 1 \cdot 14 - 0 + 1 \cdot (-4 + 3) = 13$$

EQUIVALENTLY WE CAN EXPAND ALONG ANY ROW OR COLUMN OF A 3×3 DETERMINANT:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

SO ALSO

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

e.g.

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 5 \\ -3 & -2 & 4 \end{vmatrix} = (-3)\begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= -3(-1) - (-2)(5-2) + 4(1)$$

$$= 3 + 6 + 4 = 13$$

Thus

$$\vec{a} \times \vec{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>Ex.</u> $\vec{a} = \langle 1, 2, 3 \rangle$
$\vec{b} = \langle 4, -1, 2 \rangle$

$$\vec{a} \times \vec{b} = \langle 7, 10, -9 \rangle$$

THM

For any $\vec{a} \in \mathbb{R}^3$, $\vec{a} \times \vec{a} = \vec{0}$.

THM

For any $\vec{a}, \vec{b} \in \mathbb{R}^3$,

$$\vec{a} \times \vec{b} \perp \vec{a} \quad \text{and} \quad \vec{a} \times \vec{b} \perp \vec{b}$$

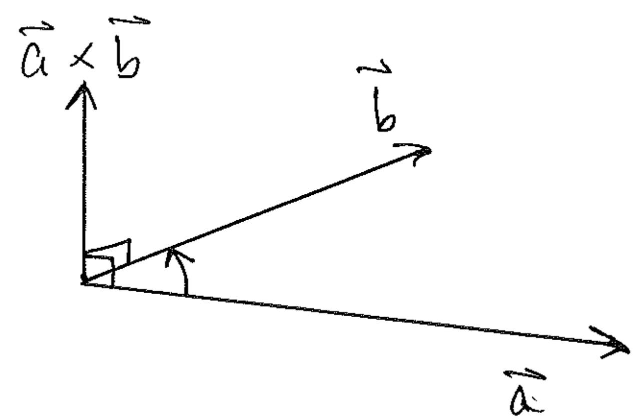
PROOF

$$\begin{aligned}
&\vec{a} \cdot (\vec{a} \times \vec{b}) \\
&= a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) \\
&= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_1 a_2 b_3 + a_1 a_3 b_2 - a_2 a_3 b_1 \\
&= 0
\end{aligned}$$

$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$ is similar. ///.

Thus $\vec{a} \times \vec{b}$ is orthogonal to a plane containing (any two representatives of) $\vec{a}$ and $\vec{b}$.

RIGHT HAND RULE



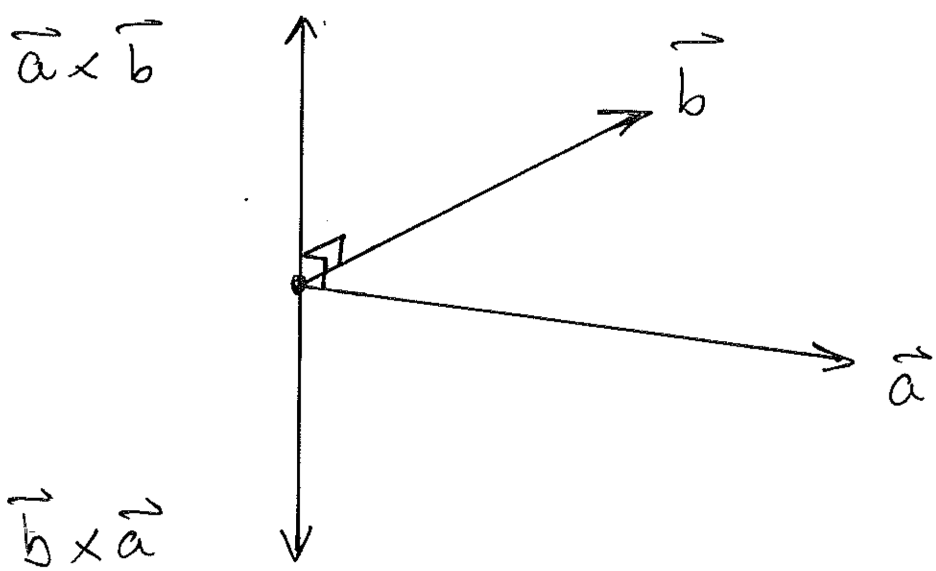
EXERCISE

PROVE THAT FOR ANY $\vec{a}, \vec{b} \in \mathbb{R}^3$

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

AND IN PARTICULAR:

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{i} = -\vec{k}$$
$$\vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{j} = -\vec{i}$$
$$\vec{k} \times \vec{i} = \vec{j} \qquad \vec{i} \times \vec{k} = -\vec{j}$$



THEOREM

LET θ BE THE ANGLE BETWEEN $\vec{a}$ AND $\vec{b}$ $(0 \le \theta \le \pi)$. THEN

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$$

PROOF:

$$|\vec{a}\times\vec{b}|^2 = (a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2$$

$= \ldots\ldots$ (EXERCISE) $\ldots\ldots$

$$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$= |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$$

$$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta.$$

HENCE

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

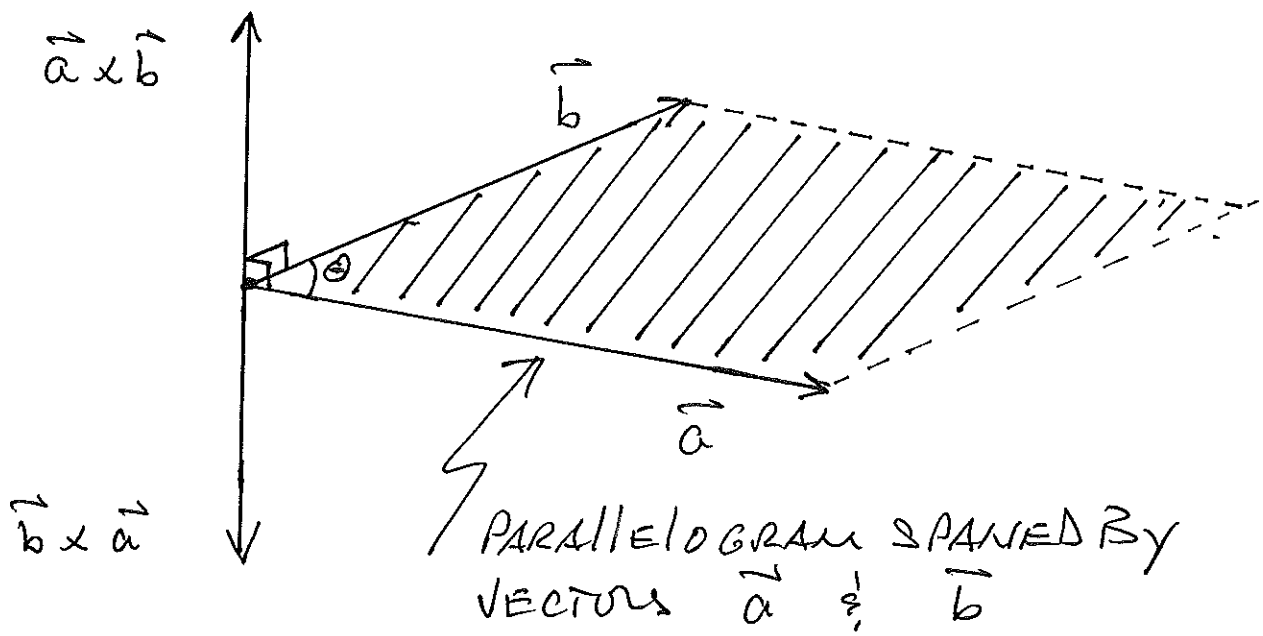
///

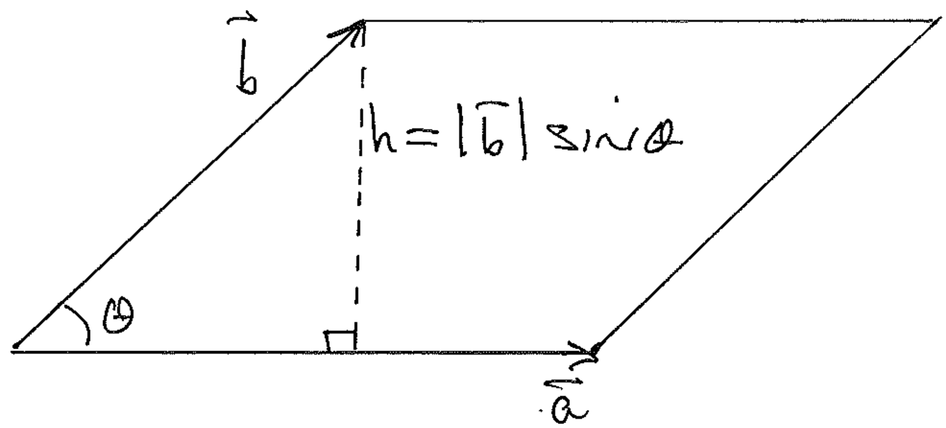
COROLLARY

TWO (NON-ZERO) VECTORS $\vec{a}$ and $\vec{b}$ ARE PARALLEL IFF

$$\vec{a}\times\vec{b} = \vec{0}$$

IF WE DRAW $\vec{a}$ AND $\vec{b}$ WITH A COMMON BASE POINT, THEN THE DEFINE A PARALLELOGRAM IN $\mathbb{R}^3$, AND $\vec{a} \times \vec{b}$ IS ORTHOGONAL TO THAT PARALLELOGRAM





OBSERVE FROM THE FIGURE THAT

$$\text{area of } \square = |\vec{a}| \cdot h = |\vec{a}||\vec{b}| \sin\theta$$

$$= |\vec{a} \times \vec{b}|$$

THM

THE AREA OF THE PARALLELOGRAM SPANED BY VECTORS $\vec{a}$ AND $\vec{b}$ IS $|\vec{a}\times\vec{b}|$.

PROPERTIES

1.) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

2.) $c(\vec{a}\times\vec{b}) = (c\vec{a})\times\vec{b} = \vec{a}\times(c\vec{b})$

3.) $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$

4.) $(\vec{a}+\vec{b})\times\vec{c} = (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c})$

5.) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

6.) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})b - (\vec{a}\cdot\vec{b})\vec{c}$

TWO MORE NOT IN BOOK!

7.) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{c}) = \vec{0}$

8.) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

EXERCISE!

PROVE (1) - (8).

BOOK DOES (5)

HINTS: (4) FOLLOWS FROM (1)-(3)

(7) FOLLOWS FROM (6).

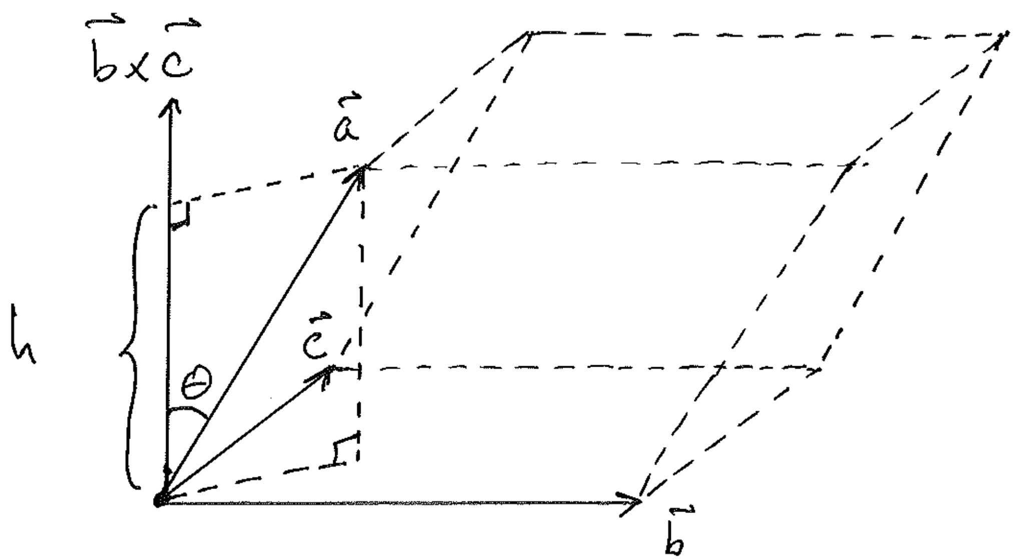
DEFN

GIVEN $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, THEIR TRIPLE PRODUCT IS : $\vec{a} \cdot (\vec{b} \times \vec{c})$

EXERCISE : SHOW THAT

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

CONSIDER THE PARALLELOPIPED SPANED BY $\vec{a}, \vec{b}, \vec{c}$



OBSERVE THAT

$$(\text{Vol of } \square) = h \cdot (\text{area of base } \square)$$
$$= (|\vec{a}| \cos\theta) \cdot |\vec{b} \times \vec{c}|$$
$$= |\vec{a}| |\vec{b} \times \vec{c}| \cos\theta$$

Thus
THE VOLUME V OF THE PARALLELOPIPED SPANED BY $\vec{a}, \vec{b}, \vec{c}$ IS GIVEN BY

$$V = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$

NOTE THERE ARE 6 ORDERS IN WHICH TO TAKE THE TRIPLE PRODUCT:

$$\underbrace{\begin{matrix} \vec{a} \cdot (\vec{b} \times \vec{c}) \\ \vec{b} \cdot (\vec{c} \times \vec{a}) \\ \vec{c} \cdot (\vec{a} \times \vec{b}) \end{matrix}}_{+\text{Vol}} \qquad \underbrace{\begin{matrix} \vec{a} \cdot (\vec{c} \times \vec{b}) \\ \vec{b} \cdot (\vec{a} \times \vec{c}) \\ \vec{c} \cdot (\vec{b} \times \vec{a}) \end{matrix}}_{-\text{Vol}}$$

READ: TORQUE P. 791-792.

The Right-Hand Rule

If $\vec{r}$ and $\vec{s}$ are not parallel, then $\vec{r}$ and $\vec{s}$ span a plane. The direction of $\vec{r} \times \vec{s}$ is taken to be perpendicular to this plane. That restricts us to two possible directions. The choice of which of these two we take is determined by the spin convention described previously which is referred to as the *right-hand rule* and can be described in several different but equivalent ways.

Imagine a standard (right-handed) screw through the origin, perpendicular to the plane spanned by $\vec{r}$ and $\vec{s}$. If you turn this screw from $\vec{r}$ to $\vec{s}$, then the direction of its resulting movement (either in to or out from the plane) is the direction of $\vec{r} \times \vec{s}$ (Figure 2.7).

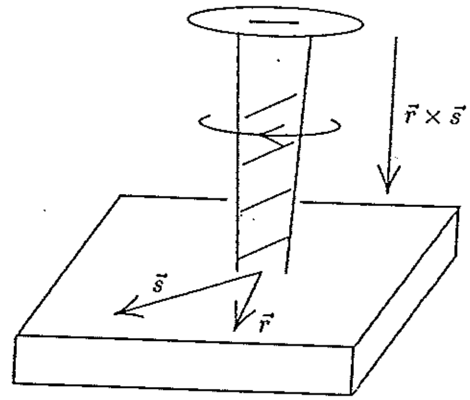


FIGURE 2.7. The right-hand rule: first version.

OR

Cup your right-hand with the thumb outstretched, as if you are about to hitchhike. Place it so that the little finger lies flat on the plane spanned by $\vec{r}$ and $\vec{s}$ and so that the direction from $\vec{r}$ to $\vec{s}$ is the same as the direction of the curl of your fingers as you travel from the wrist around to the finger tips. The thumb is pointing in the direction of $\vec{r} \times \vec{s}$ (Figure 2.8).

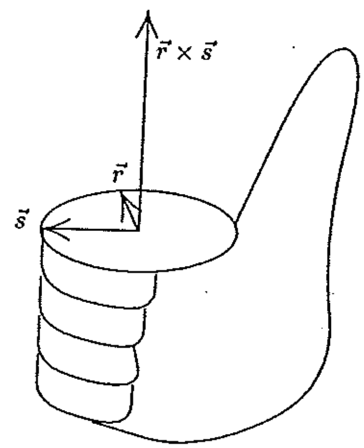


FIGURE 2.8. The right-hand rule: second version.

OR

Stretch the thumb, index finger, and middle finger of your right-hand so that each is pointing in a different direction. If you line up the thumb with $\vec{r}$ and the index finger with $\vec{s}$, then the middle finger points in the direction of $\vec{r} \times \vec{s}$ (Figure 2.9).

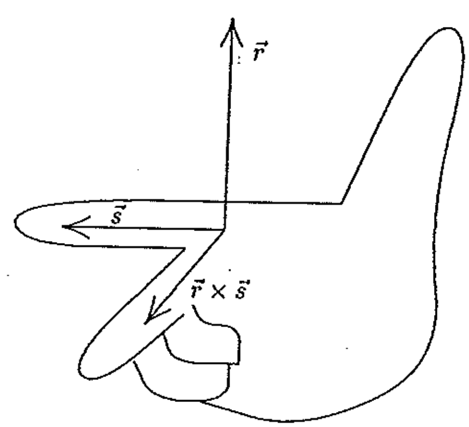


FIGURE 2.9. The right-hand rule: third version.