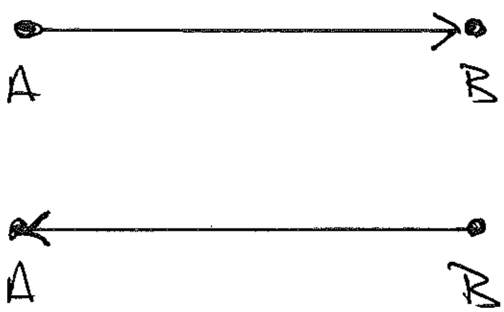
(12.2) Vectors

Let A, B be two points (in $\mathbb{R}^2$ or $\mathbb{R}^3$). The line segment AB is said to be <u>directed</u> when one of its ends is designated as the <u>initial point</u>, and the other as the <u>terminal point</u>.

<u>Notation</u>

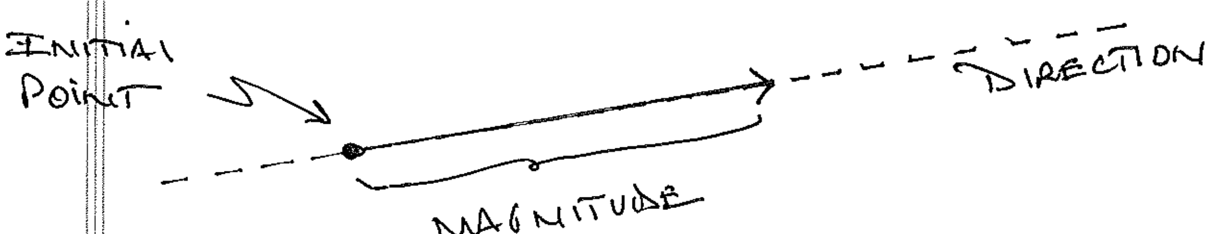


$\overrightarrow{AB}$

$\overrightarrow{BA}$

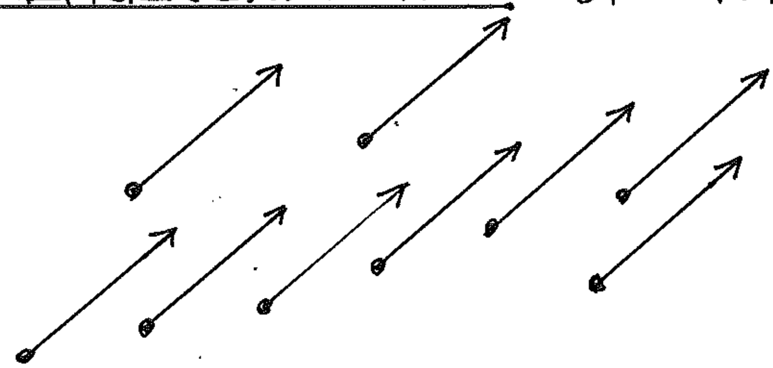
Two possible orientations of AB.

A directed line segment can be specified by giving an initial point, a magnitude (or length), and a direction.



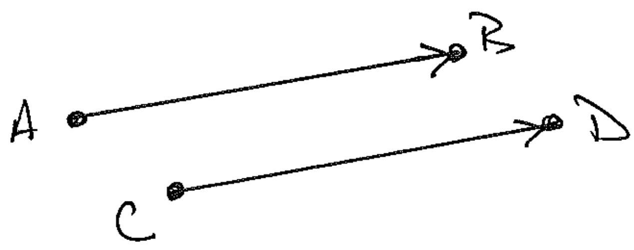
DEFN
A VECTOR (IN $\mathbb{R}^2$ OR $\mathbb{R}^3$) IS THE SET OF ALL DIRECTED LINE SEGMENTS HAVING A FIXED MAGNITUDE AND DIRECTION (BUT DIFFERENT INITIAL POINTS.)

EACH DIRECTED LINE SEGMENT IN THIS SET IS SAID TO BE A REPRESENTATIVE OF THE VECTOR.



DIFFERENT REPRESENTATIVES OF THE SAME VECTOR

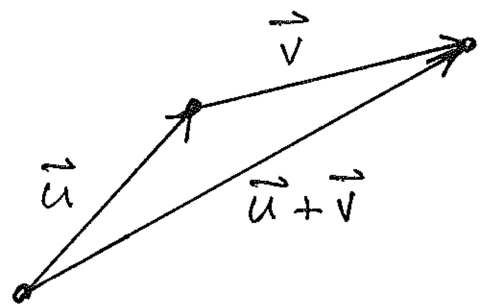
NOTATION: $\vec{V} = \overrightarrow{AB}$ THE VECTOR REPRESENTED BY THE DIR. LINE SEG. $\overrightarrow{AB}$.



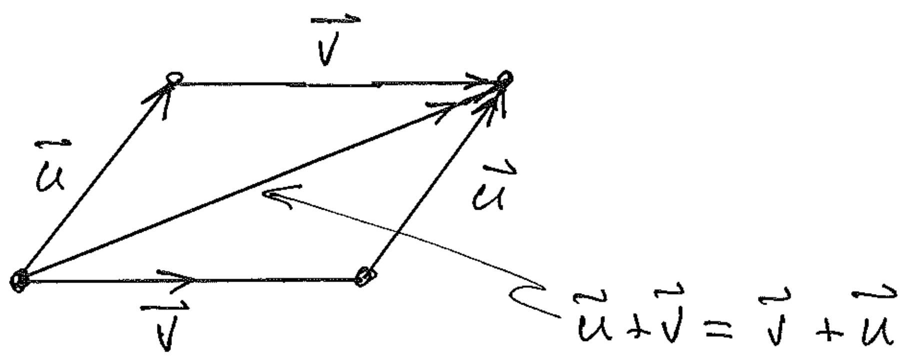
ONE VECTOR, TWO DIR. LINE SEGMENTS.

$\vec{V} = \overrightarrow{AB} = \overrightarrow{CD}$

<u>DEFN</u> (VECTOR ADDITION)
LET $\vec{u}$ AND $\vec{v}$ BE VECTORS (IN $\mathbb{R}^2$ OR $\mathbb{R}^3$) REPRESENTED SO THAT THE TERMINAL POINT OF $\vec{u}$ COINCIDES WITH THE INITIAL POINT OF $\vec{v}$. THEN THEIR SUM $\vec{u}+\vec{v}$ IS THE VECTOR REPRESENTED BY THE LINE SEGMENT FROM THE INITIAL POINT OF $\vec{u}$ TO THE TERMINAL POINT OF $\vec{v}$.

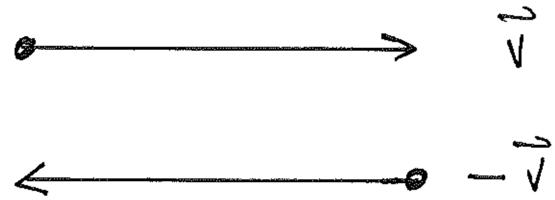


NOTICE THAT $\vec{u}+\vec{v} = \vec{v}+\vec{u}$!



THIS FACT IS SOMETIMES CALLED THE <u>PARALLELOGRAM LAW</u>.

IF $\vec{V}$ IS A VECTOR, ITS <u>NEGATIVE</u>, DENOTED $-\vec{V}$ IS REPRESENTED BY ANY SEGMENT HAVING THE SAME LENGTH AND OPPOSITE ORIENTATION.



OBSERVE THAT $-(-\vec{V}) = \vec{V}$

THE <u>ZERO VECTOR</u> $\vec{O}$ IS A SPECIAL OBJECT WITH LENGTH O AND DIRECTION UNDEFINED.

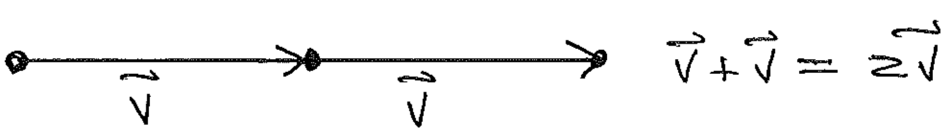
OBSERVE THAT FOR ANY VECTOR $\vec{V}$ (IN $\mathbb{R}^2$ OR $\mathbb{R}^3$):

$$\vec{V} + (-\vec{V}) = \vec{O}$$

AND

$$\vec{V} + \vec{O} = \vec{O} + \vec{V} = \vec{V} .$$

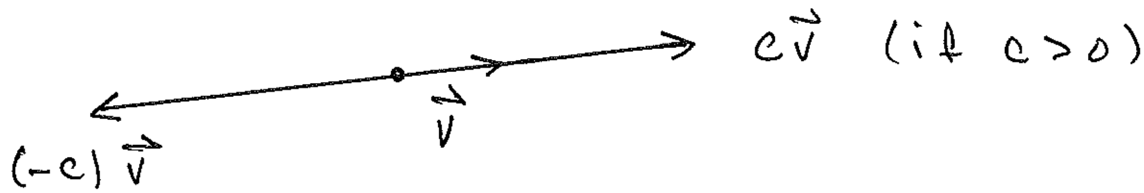
OBSERVE THAT $\vec{V}+\vec{V}$ IS A VECTOR HAVING THE SAME DIRECTION AS $\vec{V}$ AND TWICE THE MAGNITUDE.



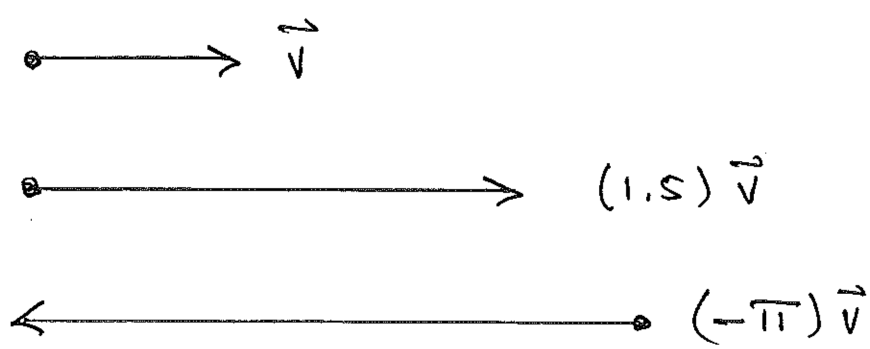
DEFN (SCALAR MULTIPLICATION)
LET $\vec{v}$ BE A VECTOR (IN $\mathbb{R}^2$ OR $\mathbb{R}^3$) AND LET $c \in \mathbb{R}$ (ALSO CALLED A SCALAR.) THE SCALAR PRODUCT $c\vec{v}$ IS THE VECTOR HAVING LENGTH $|c| \cdot$ (length of $\vec{v}$), AND DIRECTION:

$$\begin{cases} \text{SAME AS } \vec{v} & \text{IF } c > 0 \\ \text{OPPOSITE } \vec{v} & \text{IF } c < 0 \end{cases}.$$

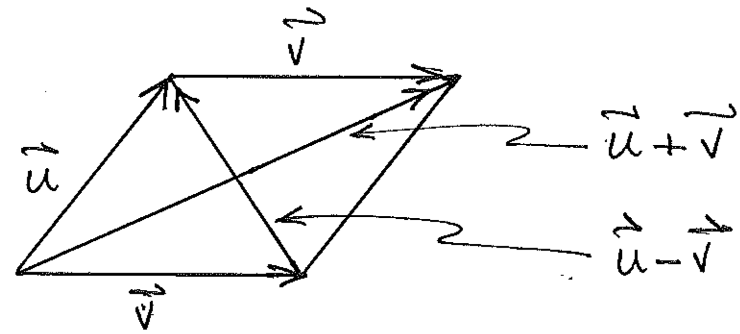
IF $c = 0$ OR $\vec{v} = \vec{0}$, THEN $c\vec{v} = \vec{0}$.



EX.



OBSERVE THAT THE PARALLELOGRAM LAW ALSO SHOWS US HOW TO SUBTRACT VECTORS.



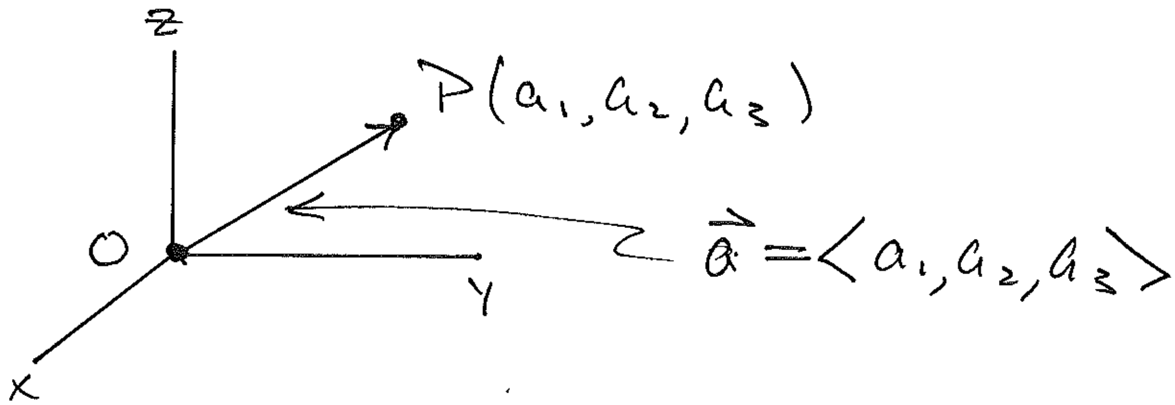
$\vec{u}-\vec{v}$ MUST BE THE VECTOR WHICH WHEN ADDED TO $\vec{v}$ GIVES $\vec{u}$:

$$\vec{v} + (\vec{u}-\vec{v}) = \vec{u}$$

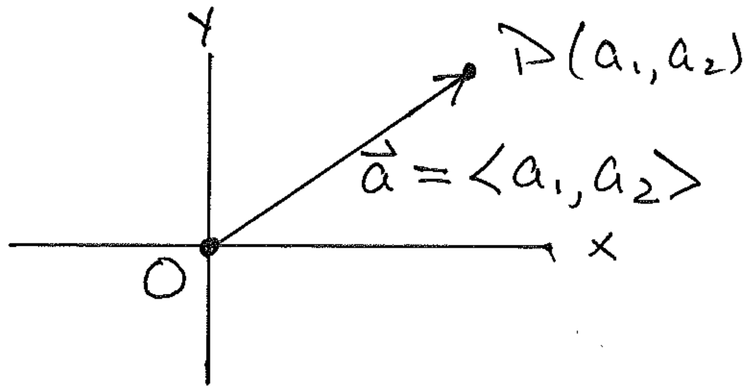
<u>DEFN</u>

LET $\vec{a}$ BE A VECTOR IN $\mathbb{R}^3$. CONSIDER THE REPRESENTATIVE OF $\vec{a}$ HAVING INITIAL POINT THE ORIGIN $O(0,0,0)$ AND TERMINAL POINT $P(a_1, a_2, a_3)$ THE <u>COMPONENTS</u> OF $\vec{a}$ ARE THE COORDINATES OF $P(a_1, a_2, a_3)$.

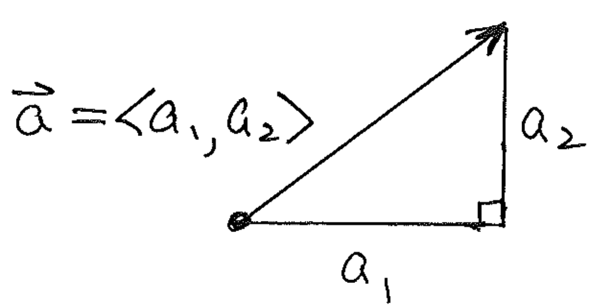
NOTATION: $\vec{a} = \langle a_1, a_2, a_3 \rangle$



THE PICTURE IS SIMILAR FOR VECTORS IN $\mathbb{R}^2$.



AN EQUIVALENT WAY TO DEFINE THE COMPONENTS OF A VECTOR $\vec{a}$ IN $\mathbb{R}^2$ WOULD BE TO DRAW $\vec{a}$ ANYWHERE IN THE PLANE, AND CONSIDER A RIGHT TRIANGLE WITH $\vec{a}$ AS ITS HYPOTENUSE.



THE LENGTH $|\vec{a}|$ OF A VECTOR $\vec{a}$ (IN $\mathbb{R}^2$ or $\mathbb{R}^3$) is EASILY GIVEN IN TERMS OF COMPONENTS BY THE DISTANCE FORMULA

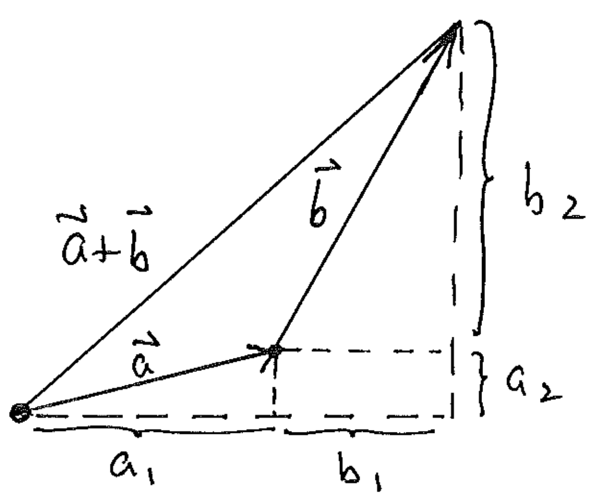
- $\vec{a} = \langle a_1, a_2 \rangle \in \mathbb{R}^2$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$

- $\vec{a} = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

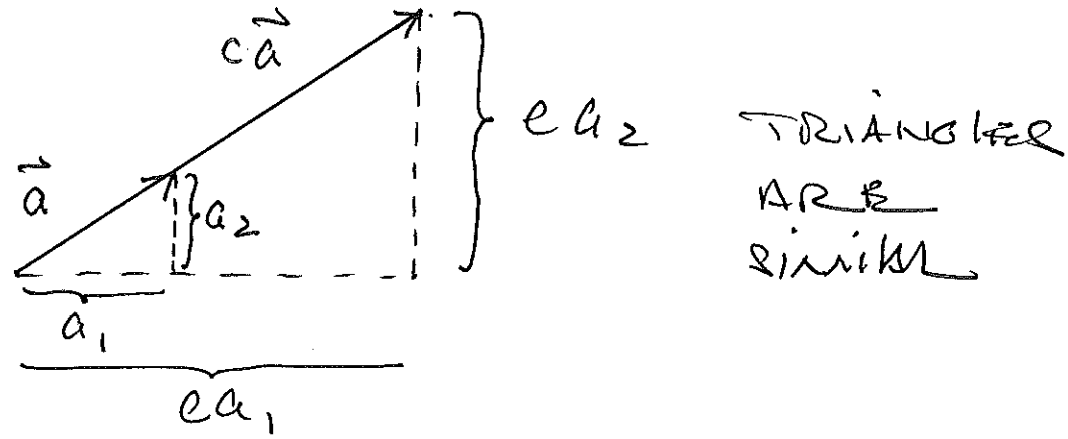
WE CAN USE COMPONENTS TO ADD VECTORS!



IF $\vec{a} = \langle a_1, a_2 \rangle$ AND $\vec{b} = \langle b_1, b_2 \rangle$ THEN

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

WE CAN ALSO COMPUTE THE SCALAR MULTIPLE $c\vec{a}$ USING COMPONENTS:



$\therefore$ IF $\vec{a} = \langle a_1, a_2 \rangle$ THEN

$$c\vec{a} = \langle ca_1, ca_2 \rangle$$

SIMILAR PICTURES COULD BE DRAWN IN $\mathbb{R}^3$, SO THAT IF

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$
$$b = \langle b_1, b_2, b_3 \rangle$$

THEN

$$\vec{a}+\vec{b} = \langle a_1+b_1, a_2+b_2, a_3+b_3 \rangle$$

AND

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle .$$

ALGEBRAIC PROPERTIES OF VECTORS:

1.) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2.) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
3.) $\vec{a}+\vec{0}=\vec{a}$
4.) $\vec{a}+(-\vec{a})=\vec{0}$
5.) $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
6.) $(c+d)\vec{a}=c\vec{a}+d\vec{a}$
7.) $(cd)\vec{a}=c(d\vec{a})$
8.) $1\vec{a}=\vec{a}$

ALL THESE PROPERTIES CAN BE PROVED USING COMPONENT REPRESENTATION. FOR INSTANCE, TO PROVE (5):

$$
\begin{aligned}
c(\vec{a}+\vec{b}) &= c\left(\langle a_1,a_2,a_3\rangle+\langle b_1,b_2,b_3\rangle\right)\\
&= c\langle a_1+b_1,\ a_2+b_2,\ a_3+b_3\rangle\\
&= \langle c(a_1+b_1),\ c(a_2+b_2),\ c(a_3+b_3)\rangle\\
&= \langle ca_1+cb_1,\ ca_2+cb_2,\ ca_3+cb_3\rangle\\
&= \langle ca_1,ca_2,ca_3\rangle+\langle cb_1,cb_2,cb_3\rangle\\
&= c\langle a_1,a_2,a_3\rangle+c\langle b_1,b_2,b_3\rangle\\
&= c\vec{a}+c\vec{b}
\end{aligned}
$$

EXERCISE PROVE OTHER PROPERTIES IN A SIMILAR MANNER.

EXERCISE: DRAW PICTURES ILLUSTRATING THESE LAWS, e.g. FIG. 16 ON P. 774.

<u>REMARK</u>: IT MAY SEEM AS IF A POINT WITH COORDINATES (a_1, a_2, a_3) AND A VECTOR WITH COMPONENTS $\langle a_1, a_2, a_3 \rangle$ ARE THE SAME THING. INDEED, IT IS OFTEN USEFUL TO IDENTIFY POINTS AND VECTORS, BUT THEY ARE NOT THE SAME AS GEOMETRIC OBJECTS.

<u>DEFN</u> A <u>UNIT VECTOR</u> IS A VECTOR WHOSE LENGTH IS 1, i.e.

$$|\vec{a}| = 1$$

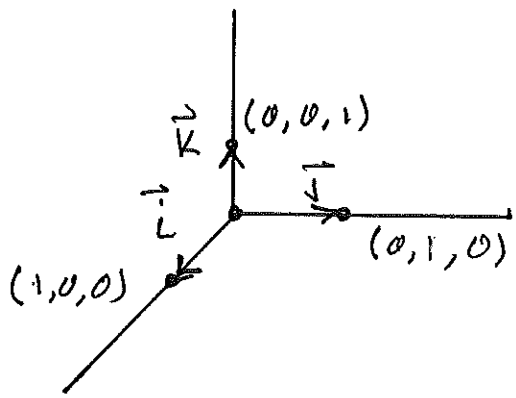
i.e.

$$a_1^2 + a_2^2 + a_3^2 = 1$$

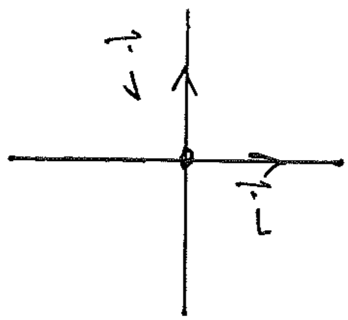
<u>DEFN</u>:
THE <u>STANDARD BASIS VECTORS</u> $\vec{i}, \vec{j}$, AND $\vec{k}$ ARE UNIT VECTORS IN THE DIRECTIONS OF THE (POSITIVE) X, Y, AND Z-AXES, RESPECTIVELY

THUS

$$\vec{i} = \langle 1, a, 0 \rangle$$
$$\vec{j} = \langle 0, 1, 0 \rangle$$
$$\vec{k} = \langle 0, 0, 1 \rangle$$



SIMILARLY IN $\mathbb{R}^2$



$\vec{i} = \langle 1, 0 \rangle$
$\vec{j} = \langle 0, 1 \rangle$

ANY VECTOR CAN BE WRITTEN IN TERMS OF THE STANDARD BASIS

$$\begin{aligned}\vec{a} &= \langle a_1, a_2, a_3 \rangle \\ &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \\ &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}\end{aligned}$$

AND SIMILARLY IN $\mathbb{R}^2$.

Ex. $\langle 2, -5, 8 \rangle = 2\vec{i} - 5\vec{j} + 8\vec{k}$

OUR DEFINITION OF SCALAR MULTIPLICATION IMPLIES THAT

$$|c\vec{a}| = |c| \cdot |\vec{a}|$$

EXERCISE: PROVE THIS USING THE COMPONENT REPRESENTATION.

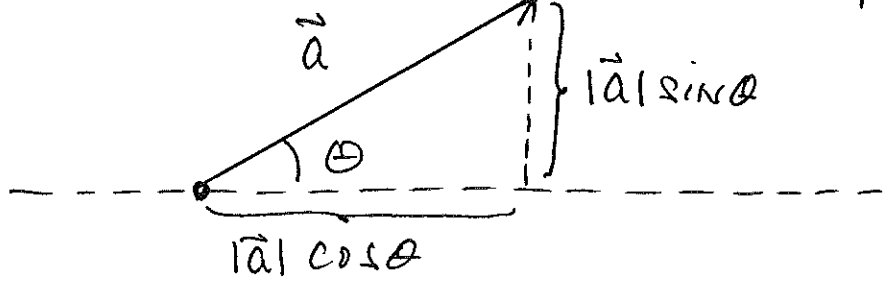
GIVEN A VECTOR $\vec{a}$, THE UNIT VECTOR $\vec{u}$ IN THE DIRECTION $\vec{a}$ IS

$$\vec{u} = \frac{1}{|\vec{a}|} \cdot \vec{a} = \frac{\vec{a}}{|\vec{a}|}$$

(VERIFY THIS.)

OBSERVE THAT A VECTOR IN $\mathbb{R}^2$ CAN BE SPECIFIED BY GIVING ITS LENGTH AND ITS ANGLE WITH A HORIZONTAL LINE

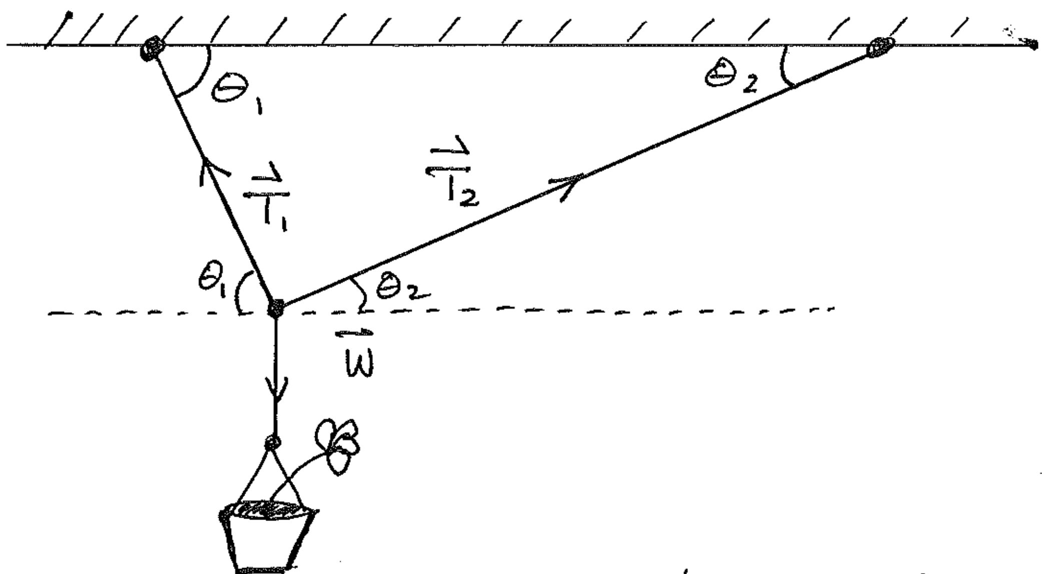
SPECIFY: $|\vec{a}|$, θ



$$\therefore \quad \vec{a} = \langle |\vec{a}| \cos\theta, |\vec{a}| \sin\theta \rangle$$

Ex.

A POTTED PLANT OF MASS 10 Kg. IS TO BE HUNG USING THE WIRE ARRANGEMENT BELOW.



FIND THE TENSIONS $\vec{T}_1$ and $\vec{T}_2$ IN THE TWO SUPPORT WIRES.

SOLUTION:

THE FORCE $\vec{W}$ DUE TO GRAVITY IS DIRECTED DOWNWARD, WITH MAGNITUDE

$$|\vec{W}| = m \cdot g = (10\ \text{Kg}) \cdot \left(9.8\ \frac{m}{s^2}\right) = 98\ N$$

(WHICH IS ABOUT 22 lbs.) THUS

$$\vec{W} = \langle 0, -98 \rangle$$

FROM THE FIGURE WE HAVE

$$\vec{T}_1 = \langle -|\vec{T}_1| \cos\theta_1 , |\vec{T}_1| \sin\theta_1 \rangle$$

$$\vec{T}_2 = \langle |\vec{T}_2| \cos\theta_2 , |\vec{T}_2| \sin\theta_2 \rangle$$

SINCE THE PLANT DOES NOT MOVE WE HAVE

$$\vec{T}_1 + \vec{T}_2 + \vec{w} = \vec{0}$$

WHENCE.

$$\begin{cases} -|\vec{T}_1| \cos\theta_1 + |\vec{T}_2| \cos\theta_2 = 0 \\ |\vec{T}_1| \sin\theta_1 + |\vec{T}_2| \sin\theta_2 = 98 \end{cases}$$

SOLVING FOR THE TWO UNKNOWNS $|\vec{T}_1|$ AND $|\vec{T}_2|$ WE HAVE

$$|\vec{T}_1| = \frac{98 \cos\theta_2}{\cos\theta_2 \cdot \sin\theta_1 + \cos\theta_1 \cdot \sin\theta_2}$$

$$|\vec{T}_2| = \frac{98 \cos\theta_1}{\cos\theta_2 \cdot \sin\theta_1 + \cos\theta_1 \cdot \sin\theta_2}$$