7.1 EIGENVALUES & EIGENVECTORS

DEFN:
LET $A \in M_n$. A NON-ZERO VECTOR $\vec{v} \in \mathbb{R}^n$ IS CALLED AN <u>EIGENVECTOR</u> OF A IFF

$$A\vec{v} = \lambda \vec{v}$$

FOR SOME $\lambda \in \mathbb{R}$. WE CALL λ THE <u>EIGENVALUE</u> ASSOCIATED WITH $\vec{v}$. WE ALSO CALL $\vec{v}$ THE EIGENVECTOR BELONGING TO λ.

RMK
THE ONE DIMENSIONAL SUBSPACE $\text{span}(\vec{v}) \subseteq \mathbb{R}^n$ IS INVARIANT UNDER $T_A : \mathbb{R}^n \to \mathbb{R}^n$. i.e. IF $\vec{u} \in \text{span}(\vec{v})$ THEN $A\vec{u} \in \text{span}(\vec{v})$. TO SEE THIS SUPPOSE $\vec{u} = \alpha \vec{v}$. THEN

$$\begin{aligned} A\vec{u} &= A(\alpha\vec{v}) = \alpha(A\vec{v}) = \alpha(\lambda\vec{v}) \\ &= \lambda(\alpha\vec{v}) = \lambda\vec{u}. \end{aligned}$$

THIS CALCULATION SHOWS THAT ANY SCALAR MULTIPLE OF AN EIGENVECTOR IS ANOTHER EIGENVECTOR BELONGING TO THE SAME EIGENVALUE.

SIMILARLY IF $\vec{v}_1, \vec{v}_2$ ARE TWO EIGENVECTORS OF A BELONGING TO THE SAME EIGENVALUE λ, THEN THEIR SUM $\vec{v}_1 + \vec{v}_2$ IS ALSO AN EIGENVECTOR BELONGING TO λ !

$$\begin{aligned} A(\vec{v}_1 + \vec{v}_2) &= A\vec{v}_1 + A\vec{v}_2 \\ &= \lambda\vec{v}_1 + \lambda\vec{v}_2 \\ &= \lambda(\vec{v}_1 + \vec{v}_2) \end{aligned}$$

THUS THE SET OF ALL EIGENVECTORS OF A BELONGING TO A FIXED EIGENVALUE λ FORMS AN (INVARIANT) SUBSPACE OF $\mathbb{R}^n$.

EX.

$$A = \begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \lambda = 2$$

EX

$$A = \begin{pmatrix} 5 & -3 \\ 7 & -5 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_1 = 2$$

$$\vec{v}_2 = \begin{pmatrix} 3 \\ 7 \end{pmatrix}, \quad \lambda_2 = -2$$

SUPPOSE 0 IS AN EIGENVALUE OF $A \in M_n$. THEN $A\vec{v} = 0\vec{v} = \vec{0}$ FOR SOME NON-ZERO VECTOR $\vec{v}$. $\therefore$ $\ker(A) \neq \{\vec{0}\}$. CONVERSELEY IF $\ker(A) = \{\vec{0}\}$, THEN THERE IS NO NON-ZERO VECTOR $\vec{v}$ SUCH THAT $A\vec{v} = \vec{0}$, AND 0 IS NOT AN EIGENVALUE OF A.

THEREFORE 0 IS AN EIGENVALUE OF A IFF $\ker(A) \neq \{\vec{0}\}$.

THEOREM

LET $A \in M_n$. T.F.A.E.

(1) A IS INVERTIBLE

(2) $A\vec{x} = \vec{b}$ HAS A UNIQUE SOLUTION FOR ALL $\vec{b} \in \mathbb{R}^n$

(3) $\mathrm{RREF}(A) = I_n$

(4) $\mathrm{rank}(A) = n$ $(\mathrm{nullity}(A) = 0)$

(5) $\mathrm{im}(A) = \mathbb{R}^n$

(6) $\ker(A) = \{0\}$

(7) COLUMNS OF A FORM A BASIS OF $\mathbb{R}^n$

(8) " " " SPAN $\mathbb{R}^n$

(9) " " " ARE LINEARLY INDEPENDENT

(10) $\det(A) \neq 0$

(11) 0 IS NOT AN EIGENVALUE OF A

HW (7.1) 4, 8, 10, 12, 14, 34, 36, 38, 40