6.2 Properties of Determinants

The Laplace expansion is a highly inefficient method for computing a determinant.

<u>Theorem</u>
Let $B \in M_n$ be the result of performing one of the following elementary row operations on $A \in M_n$.

<u>ERO 1</u>: Multiply some row of A by $\alpha \in \mathbb{R}$.
Then $\det B = \alpha \cdot \det A$

<u>ERO 2</u>: Swap two rows of A.
Then $\det B = -\det A$

<u>ERO 3</u>: Add a multiple of one row of A to another row.
Then $\det B = \det A$.

Recall that we proved this for 2×2 matrices (p. 60 of these notes.) We omit the general proof.

<u>Exercise</u>
Prove this theorem directly for 3×3 matrices. Use the formula

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei - afh + bfg - bdi + cdh - ceg$$

THEOREM

SUPPOSE THAT, STARTING WITH $A \in M_n$, ONE PERFORMS A SEQUENCE OF EROs TO REACH A MATRIX $B \in M_n$. SPECIFICALLY SUPPOSE

- ERO 1 IS PERFORMED r TIMES: DIVIDE BY THE (NON-ZERO) CONSTANTS $k_1, \cdots k_r$
- ERO 2 IS PERFORMED s TIMES
- ERO 3 IS PERFORMED ANY NUMBER OF TIMES.

THEN

$$\det(A) = (-1)^s \cdot k_1 \cdot k_2 \cdots k_r \cdot \det(B)$$

PROOF:

BY REPEATED APPLICATION OF THE PREVIOUS THEOREM, WE HAVE

$$\det(B) = (-1)^s \cdot \frac{1}{k_1} \cdot \frac{1}{k_2} \cdots \frac{1}{k_r} \cdot \det(A).$$

MULTIPLY THIS EQUATION BY $(-1) \cdot k_1 \cdots k_r$ TO OBTAIN THE RESULT.

///.

RMK IF THE SEQUENCE OF EROs ARE SPECIFIED BY GAUSS-JORDAN ELIMINATION THEN $B = RREF(A)$, $r = \text{rank}(A)$, AND

$$\det(A) = (-1)^s \cdot k_1 k_2 \cdots k_r \det(RREF(A)).$$

(NOTE: SOME OF THE k_i MAY $= 1$.)

<u>Ex.</u> $\det(A) = (-8)\cdot\left(-\frac{27}{8}\right) = \boxed{27}$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -1 & 3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & -7 \\ 0 & 5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{7}{8} \\ 0 & 5 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 5/4 \\ 0 & 1 & 7/8 \\ 0 & 0 & -27/8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5/4 \\ 0 & 1 & 7/8 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

OBSERVE THAT THIS PROCESS NEED NOT GO ALL THE WAY TO RREF(A), i.e. JUST STOP AT SOME POINT WHERE THE DETERMINANT IS EASY TO COMPUTE.

NOW RECALL

$$\det(\mathrm{RREF}(A)) = \begin{cases} \det(I_n) = 1 & \text{IF } A \text{ INVERTIBLE} \\ 0 & \text{IF } A \text{ NON-INVERTIBLE} \end{cases}$$

HENCE, UPON PERFORMING GAUSS JORDAN ELIMINATION

$$\det(A) = \begin{cases} (-1)^s k_1 \cdots k_r & \text{IF } A \text{ INVERTIBLE} \\ 0 & \text{IF } A \text{ NON-INVERTIBLE} \end{cases}$$

<u>COROLLARY</u>

$A \in M_n$ IS INVERTIBLE IF AND ONLY IF

$$\det(A) \neq 0.$$

LEMMA (6.2 PROBLEM #34)
SUPPOSE $A, B \in M_n$ AND A IS INVERTIBLE. THEN
$$\mathrm{RREF}(A \mid AB) = (I_n \mid B)$$

PROOF
SINCE A IS INVERTIBLE $\mathrm{RREF}(A) = I_n$, AND CLEARLY THE SAME EROS THAT TRANSFORM A INTO I_n ALSO TRANSFORM $(A \mid AB)$ INTO $(I_n \mid C)$ FOR SOME $C \in M_n$. SINCE $(I_n \mid C)$ IS IN RREF, WE NEED ONLY SHOW $C = B$.

OBSERVE THAT THE COLUMNS OF $\begin{pmatrix} B \\ -I_n \end{pmatrix}$ ARE IN $\mathrm{KER}(A \mid AB)$ SINCE

$$(A \mid AB) \cdot \begin{pmatrix} B \\ -I_n \end{pmatrix} = AB + AB(-I_n) = AB - AB = 0.$$

RECALL THAT THE EROS LEAVE THE KERNEL OF A MATRIX UNCHANGED, SO THE SAME COLUMNS BELONG TO $\mathrm{KER}(I_n \mid C)$. THUS

$$0 = (I_n \mid C)\begin{pmatrix} B \\ -I_n \end{pmatrix} = I_n B + C(-I_n) = B - C,$$

HENCE
$$C = B,$$

AS REQUIRED. ///.

THEOREM

LET $A, B \in M_n$. THEN $\det(AB) = \det A \cdot \det B$.

PROOF:

FIRST SUPPOSE A IS INVERTIBLE. THEN $RREF(A) = I_n$, AND BY THE LEMMA

$$* \qquad RREF(A \mid AB) = (I_n \mid B).$$

SUPPOSE THAT TO TRANSFORM A INTO I_n WE SWAP s TIMES (ERO 2) AND DIVIDE BY $k_1, k_2, \ldots, k_r$ (ERO 1). THEN

$$\det(A) = (-1)^s k_1 k_2 \cdots k_r.$$

BUT BY $*$, THE SAME EROs TRANSFORM AB INTO B, HENCE

$$\det(AB) = (-1)^s k_1 k_2 \cdots k_r \cdot \det B$$

$$= \det A \cdot \det B.$$

NOW SUPPOSE A IS NOT INVERTIBLE. THEN $\det(A) = 0$. ONE CHECKS (EXERCISE) THAT $\operatorname{im}(AB) \subseteq \operatorname{im}(A)$, WHENCE

$$\operatorname{rank}(AB) \leq \operatorname{rank}(A) < n$$

THUS ALSO AB IS NOT INVERTIBLE, AND $\det(AB) = 0$. THEREFORE

$$\det A \cdot \det B = 0 \cdot \det B = 0 = \det(AB). \qquad ///$$

COROLLARY

IF A IS INVERTIBLE THEN

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

PROOF:

WE HAVE

$$\det(A)\cdot\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1,$$

AND THE RESULT FOLLOWS. ///.

COROLLARY

IF $A \sim B$ THEN $\det A = \det B$.

PROOF:

SUPPOSE $B = S^{-1}AS$, WHERE $S \in M_n$ IS INVERTIBLE. THEN

$$\det B = \det(S^{-1})\cdot\det(A)\cdot\det(S) = \det(A)$$

SINCE $\det(S^{-1}) = (\det S)^{-1}$. ///.

THIS LAST RESULT SUGGESTS HOW WE CAN DEFINE THE DETERMINANT OF A LINEAR TRANSFORMATION $T: V \to V$, WHERE V IS ANY LINEAR SPACE.

SIMPLY PICK ANY BASIS $\mathcal{B}$ OF V, FORM THE MATRIX B OF T WITH RESPECT TO $\mathcal{B}$,

THEN DEFINE

$$\det(T) = \det(B).$$

RECALL THAT IF WE PICK ANY OTHER BASIS $\mathcal{A}$ OF V AND FORM THE $\mathcal{A}$-MATRIX A OF T, THEN $A \sim B$, WHENCE

$$\det(A) = \det(B).$$

THEREFORE THE ABOVE DEFINITION IS INDEPENDENT OF OUR CHOICE OF BASIS.

EXERCISE.

SHOW THAT

a.) $T: V \to V$ is invertible iff $\det(T) \neq 0$.

b.) IF $T_1 : V \to V$ AND $T_2 : V \to V$, THEN

$$\det(T_2 \circ T_1) = \det(T_2) \cdot \det(T_1).$$

HW (6.2): 4, 6, 8, 10, 18, 20, 34